Design of Tableau-Based Automated Theorem Provers and Production of Machine-Chekable Proofs

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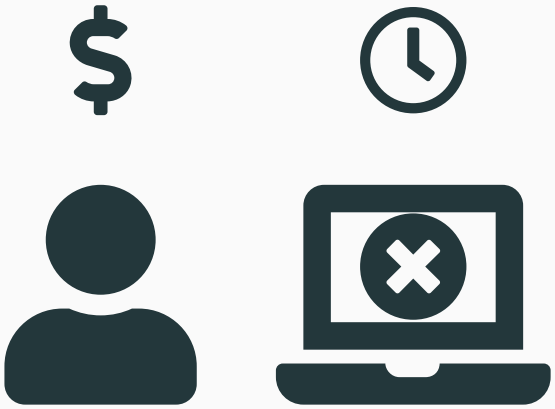
Bugs?

Tests

- Cheaper
- Faster
- Generate partial use cases and check if the system is working as expected



Formal Methods

- Expensive
- Time-consuming
- Use rigorous mathematical techniques to ensure that a system is working as expected

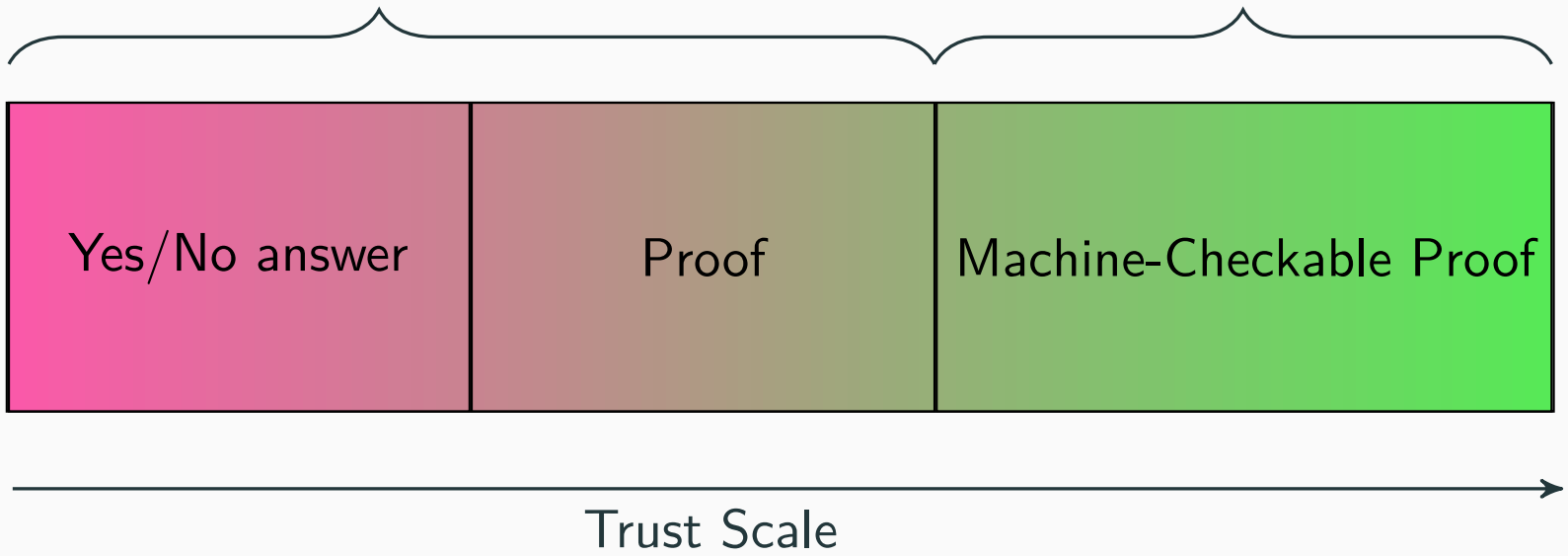
Proofs

Automated Theorem Proving

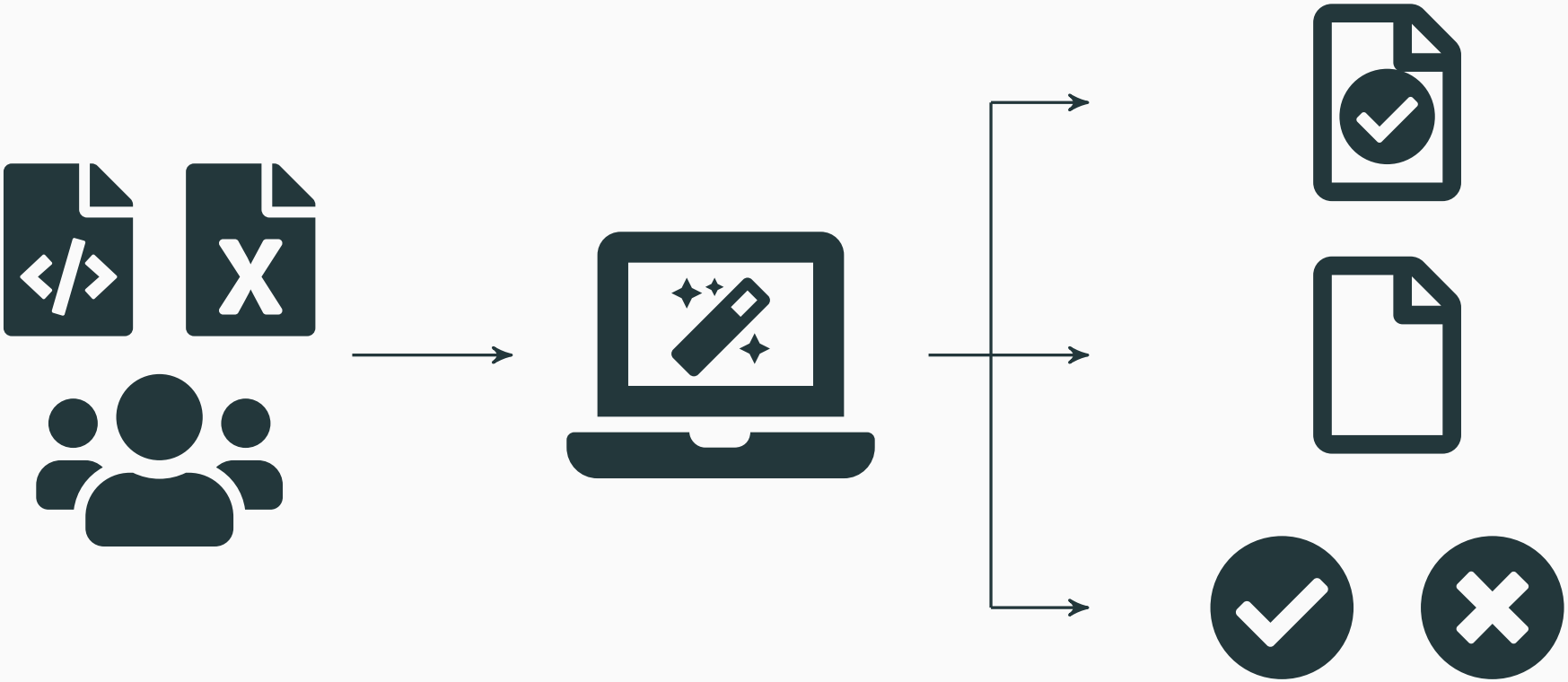
- Click-and-prove software
- Searching for a proof all by themselves
- Output a statement or a proof-like trace

Interactive Theorem Proving

- Proof assistants
- Guide humans towards proofs
- Proofs are machine-checkable and certified



Big Picture



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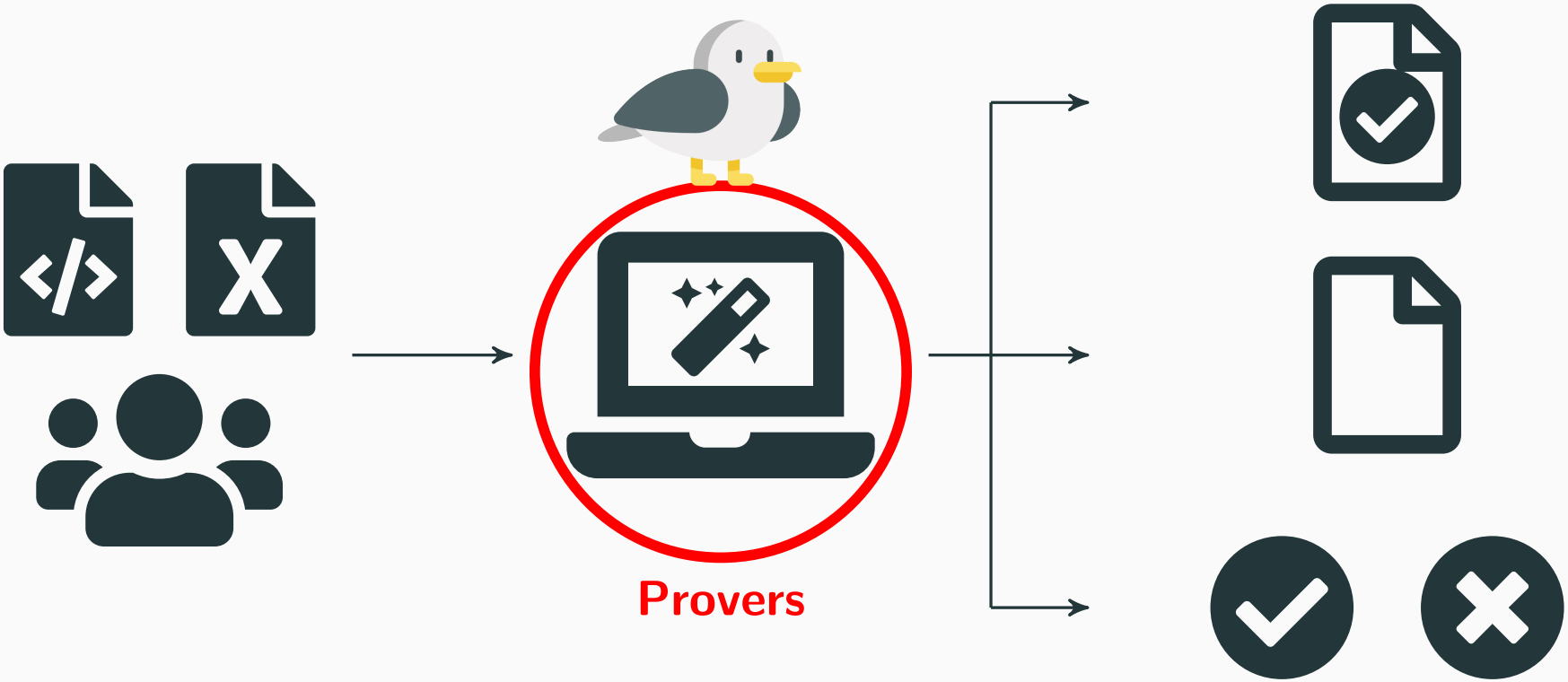

1. Preliminary Notions

Logic

What is Logic?

- Study of correct reasoning
- Mathematical representation of the world
- Truth value of a statement

First-Order Logic (FOL)

- Expressivity: elements and properties about them
- Efficient reasoning methods
- Semi-decidable

Method of Analytic Tableaux

Principle

- A set of axioms and one conjecture
- Refutation
- Syntactic rules: $\odot, \alpha, \delta, \beta, \gamma$
- Close all the branches

Origin & Strengths

- Beth and Hintikka
- Extended by Smullyan and Fitting
- Unaltered original formula
- Output a proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\, P(x) \Rightarrow (P(a) \wedge P(b)))}{\neg(P(a) \Rightarrow (P(a) \wedge P(b)))}\,\gamma_{\neg\exists}}{P(a), \neg(P(a) \wedge P(b))}\,\alpha_{\neg\Rightarrow}}{\dfrac{\neg P(a)}{\odot}\,\odot \qquad \dfrac{\neg P(b)}{\dfrac{\dfrac{\neg(P(b) \Rightarrow (P(a) \wedge P(b)))}{P(b), \neg(P(a) \wedge P(b))}\,\alpha_{\neg\Rightarrow}}{\odot}\,\odot}\,\gamma_{\neg\exists}}\,\beta_{\neg\wedge}$$

Tableau-Based Proof-Search Procedure

Rules

- $\odot$: Closure rule
- α, β: Expands the tree
- γ: Free variables
- δ: Skolemization

Tableaux in AR

- Free variables
- Substitutions (local & global)

$$\frac{Human(Socrates), \neg Human(Socrates)}{\odot}\ \odot$$

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$$Human(Socrates)$$
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$$\neg(\exists x.\ P(x) \Rightarrow (P(a) \wedge P(b)))$$

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2. Fairness Management in Tableau Proof-Search Procedure: a Concurrent Approach

Fairness in Tableaux

Fairness

A proof-search procedure is *fair* if and only if each formula on which a non-γ-rule can be applied occurs in a subsequent step, and every γ-rules will be computed an arbitrary number of times.

"At the present time, no strongly complete, destructive tableau proof procedure is known that works well in practice."
— Reiner Hähnle, *Handbook of Automated Reasoning Vol.1*, 2001

Fariness Management

Unfairness Sources

- The selection of a branch B (*select branch*)
- Determining whether B should be closed or expanded (*select mode*)
- If B is to be closed, the choice of a pair of complementary literals and thus a closing substitution (*select pair*)
- If B is to be expanded, the selection of a formula to which an expansion rule is applied (*select formula*)

State-of-the-Art Answers & Heuristics

- Limit the number of application of γ-rules
- Iterative deepening
- Rules ordering ($\odot \prec \alpha \prec \delta \prec \beta \prec \gamma$)

Select Pair and Select Mode

$$\neg P(a)$$
$$\neg Q(b)$$
$$\neg S(c)$$
$$\forall x.\ (P(x) \vee Q(x)) \wedge \forall y.\ S(x)$$

Select Pair and Select Mode

$$\dfrac{\begin{array}{c}\neg P(a)\\ \neg Q(b)\\ \neg S(c)\\ \forall x.\ (P(x) \vee Q(x)) \wedge \forall y.\ S(x)\end{array}}{P(X) \vee Q(X), \forall y.\ S(X)}\ \gamma_\forall$$

Select Pair and Select Mode

$$\dfrac{\dfrac{\begin{array}{c}\neg P(a)\\ \neg Q(b)\\ \neg S(c)\\ \forall x.\ (P(x) \vee Q(x)) \wedge \forall y.\ S(x)\end{array}}{P(X) \vee Q(X), \forall y.\ S(X)}\ \gamma_{\forall}}{P(X), \forall y.\ S(X) \qquad\qquad Q(X), \forall y.\ S(X)}\ \beta_{\Leftrightarrow}$$

Select Pair and Select Mode

$$\dfrac{\dfrac{\begin{array}{c}\neg P(a)\\ \neg Q(b)\\ \neg S(c)\\ \forall x.\ (P(x) \vee Q(x)) \wedge \forall y.\ S(x)\end{array}}{P(X) \vee Q(X), \forall y.\ S(X)}\ \gamma_{\forall}}{P(X), \forall y.\ S(X) \qquad\qquad Q(X), \forall y.\ S(X)}\ \beta_{\Leftrightarrow}$$

Select Pair and Select Mode

$$\dfrac{\dfrac{\dfrac{\neg P(a)}{\neg Q(b)}}{\neg S(c)}}{\dfrac{\dfrac{\forall x.\,(P(x) \vee Q(x)) \wedge \forall y.\,S(x)}{P(X) \vee Q(X), \forall y.\,S(X)}\,\gamma_\forall}{\dfrac{\dfrac{P(X), \forall y.\,S(X)}{\sigma = \{X \mapsto a\}}\,\odot_\sigma}{Q(X), \forall y.\,S(X)}\,\beta_\Leftrightarrow}}$$

Select Pair and Select Mode

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg P(a) \quad \neg Q(b) \quad \neg S(c) \quad \forall x.\,(P(x) \vee Q(x)) \wedge \forall y.\,S(x)}{P(a) \vee Q(a), \forall y.\,S(a)}\,\gamma_\forall}{P(a), \forall y.\,S(a)}\,\odot_\sigma \qquad Q(a), \forall y.\,S(a)}{\sigma = \{X \mapsto a\}}\,\beta_\Leftrightarrow$$

Select Pair and Select Mode

$$\neg P(a)$$
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$$\dfrac{\dfrac{\forall x.\ (P(x) \vee Q(x)) \wedge \forall y.\ S(x)}{P(a) \vee Q(a), \forall y.\ S(a)}\ \gamma\forall}{\dfrac{P(a), \forall y.\ S(a)}{\sigma = \{X \mapsto a\}}\ \odot_\sigma \qquad \dfrac{Q(a), \forall y.\ S(a)}{S(a)}\ \gamma\forall}\ \beta_\Leftrightarrow$$

Select Pair and Select Mode

$$\neg P(a)$$

$$\neg Q(b)$$

$$\neg S(c)$$

$$\dfrac{\forall x.\ (P(x) \lor Q(x)) \land \forall y.\ S(x)}{P(a) \lor Q(a), \forall y.\ S(a)}\ \gamma_\forall$$

$$\beta_\Leftrightarrow:\quad P(a), \forall y.\ S(a) \qquad\qquad Q(a), \forall y.\ S(a)$$

$$\dfrac{P(a), \forall y.\ S(a)}{\sigma = \{X \mapsto a\}}\ \odot_\sigma \qquad\qquad \dfrac{\dfrac{Q(a), \forall y.\ S(a)}{S(a)}\ \gamma_\forall}{P(X_2) \lor Q(X_2), \forall y.\ S(X_2)}\ \gamma_\forall$$

Select Pair and Select Mode

$$\neg P(a)$$
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$$\dfrac{\forall x.\ (P(x) \lor Q(x)) \land \forall y.\ S(x)}{P(a) \lor Q(a), \forall y.\ S(a)}\ \gamma_\forall$$

$$\beta_\Leftrightarrow$$

$$\dfrac{P(a), \forall y.\ S(a)}{\sigma = \{X \mapsto a\}}\ \odot_\sigma \qquad \dfrac{Q(a), \forall y.\ S(a)}{S(a)}\ \gamma_\forall$$

$$\gamma_\forall$$

$$P(X_2) \lor Q(X_2), \forall y.\ S(X_2)$$

$$\beta_\lor$$

$$P(X_2), \forall y.\ S(X_2) \qquad Q(X_2), \forall y.\ S(X_2)$$

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$$\beta_\Leftrightarrow$$

$$\dfrac{P(a), \forall y.\ S(a)}{\sigma = \{X \mapsto a\}}\ \odot_\sigma$$

$$\dfrac{Q(a), \forall y.\ S(a)}{S(a)}\ \gamma_\forall$$

$$\dfrac{S(a)}{P(X_2) \vee Q(X_2), \forall y.\ S(X_2)}\ \gamma_\forall$$

$$\dfrac{P(X_2) \vee Q(X_2), \forall y.\ S(X_2)}{P(X_2), \forall y.\ S(X_2) \qquad Q(X_2), \forall y.\ S(X_2)}\ \beta_\vee$$

$$\dfrac{P(X_2), \forall y.\ S(X_2)}{\cdots} \qquad \dfrac{Q(X_2), \forall y.\ S(X_2)}{\cdots}$$

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Select Pair and Select Mode

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg P(a)}{\neg Q(b)}}{\neg S(c)}}{\forall x.\ (P(x) \vee Q(x)) \wedge \forall y.\ S(x)}}{P(c) \vee Q(c), \forall y.\ S(c)}\ \gamma_\forall}{P(c), \forall y.\ S(c) \qquad\qquad Q(c), \forall y.\ S(c)}\ \beta_\Leftrightarrow}{\dfrac{P(c), \forall y.\ S(c)}{S(c)}\ \gamma_\forall}}{\sigma = \{X \mapsto c\}}\ \odot_\sigma$$

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Exploring Branches in Parallel?

Approach

- Each branch searches for a local solution
- Management of multiple solutions with successive attempts and backtracking
- Forbid previously tried solutions
- Iterative deepening, limit of γ-rule and rules ordering

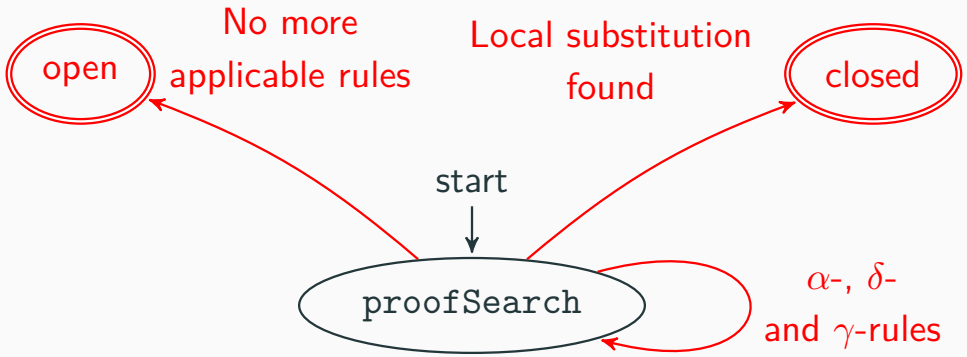
New Challenges

- Free variable dependency
- Communication between branches

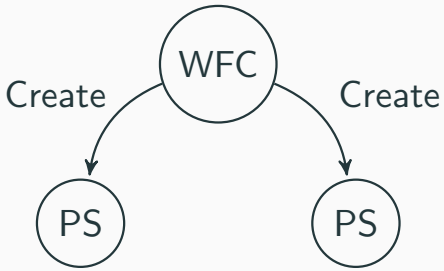
Procedures Interactions

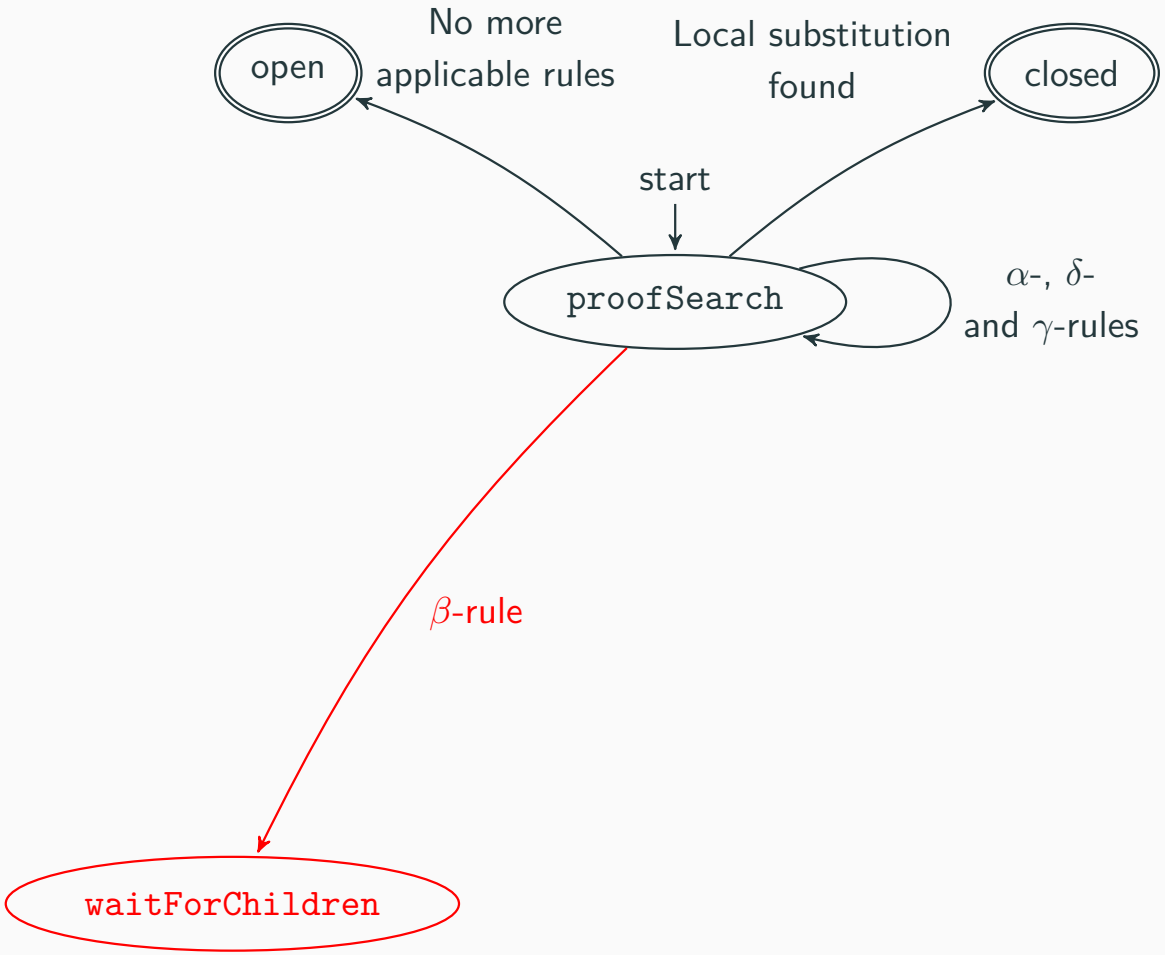
Procedures Interactions

PS



Procedures Interactions





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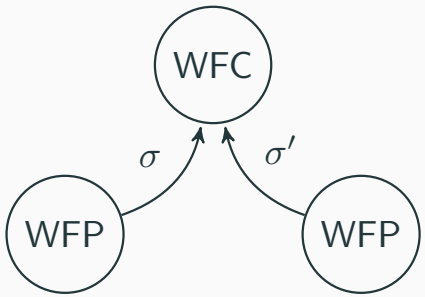

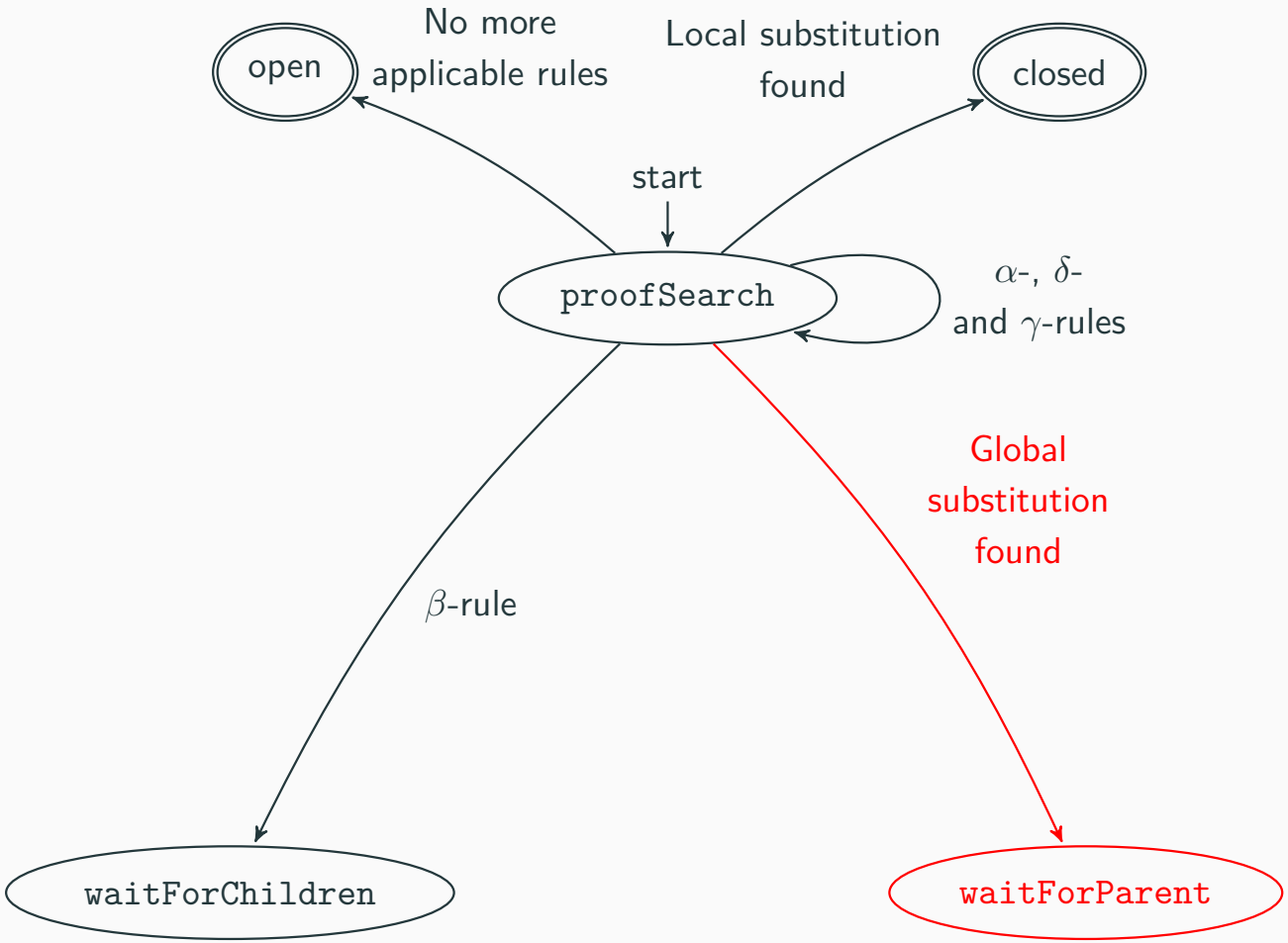

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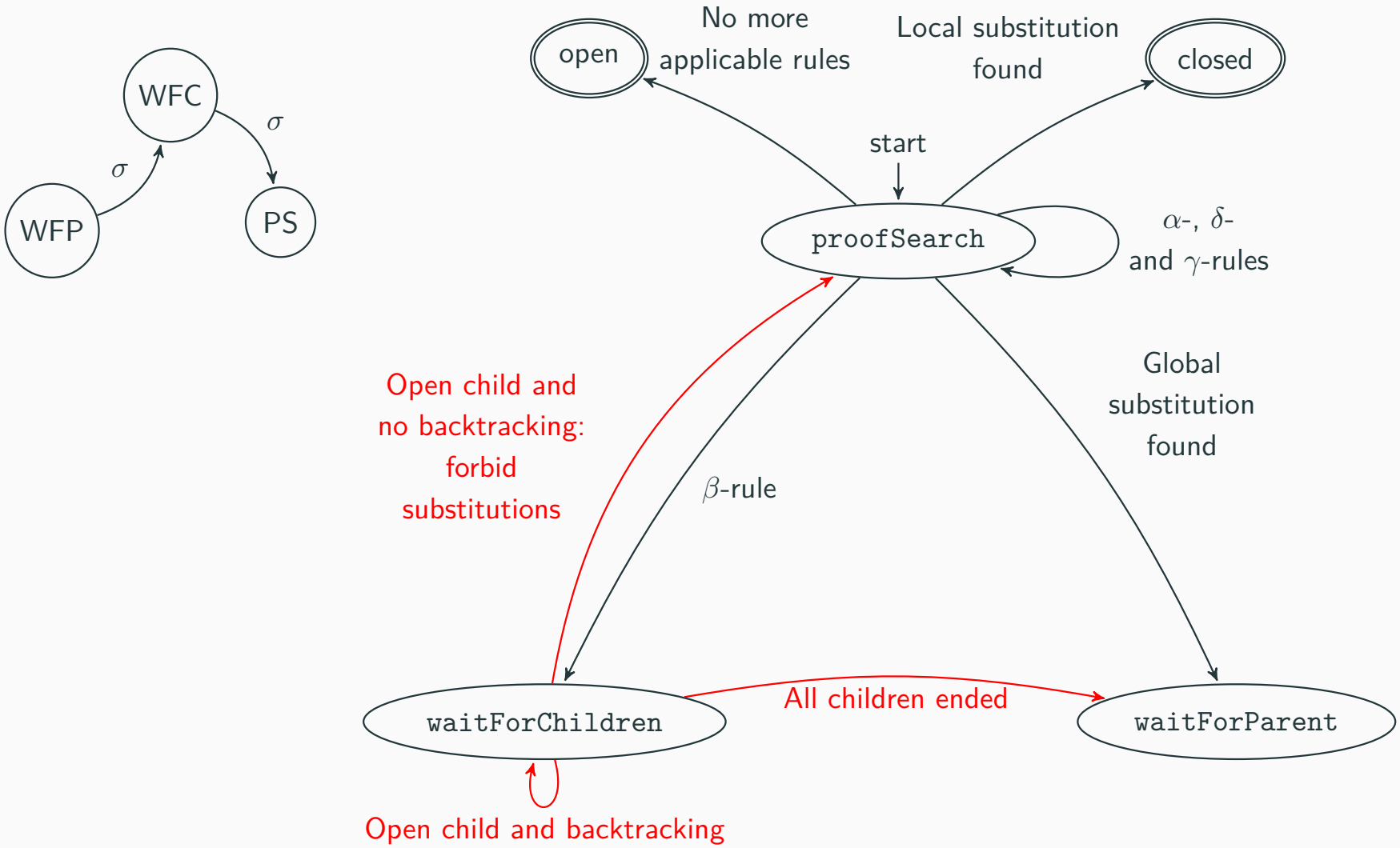

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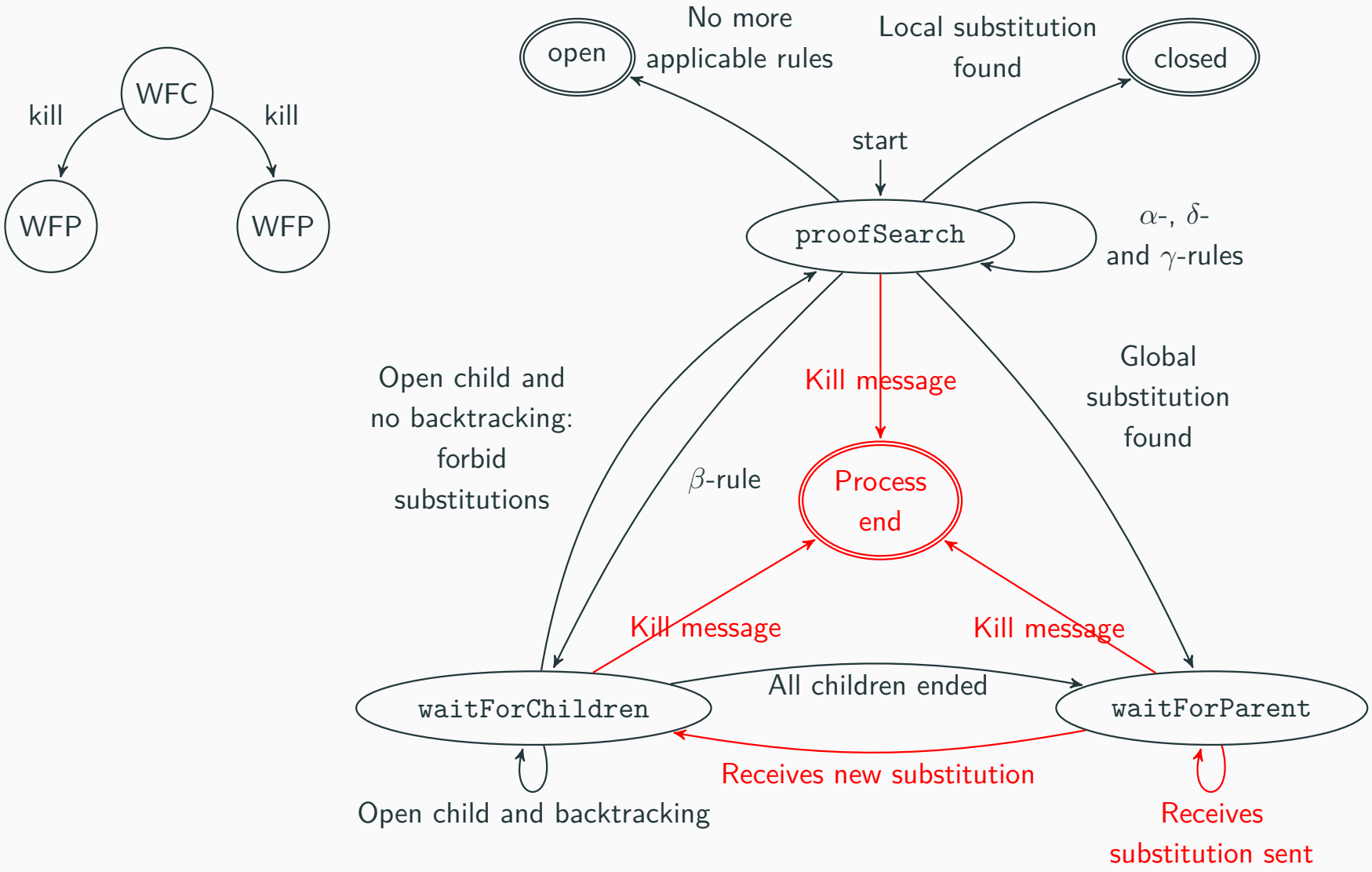

Solving Fairness Issues with Concurrency

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$$\dfrac{\begin{array}{c}\neg P(a)\\ \neg Q(b)\\ \neg S(c)\\ \forall x.\ (P(x) \vee Q(x)) \wedge \forall y.\ S(x)\end{array}}{P(X) \vee Q(X), \forall y.\ S(X)}\ \gamma_\forall$$

Solving Fairness Issues with Concurrency

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\neg P(a)\\ \neg Q(b)\\ \neg S(c)\\ \forall x.\ (P(x) \vee Q(x)) \wedge \forall y.\ S(x)\end{array}}{P(X) \vee Q(X), \forall y.\ S(X)}\,\gamma_\forall}{P(X), \forall y.\ S(X) \qquad Q(X), \forall y.\ S(X)}\,\beta_\Leftrightarrow$$

Solving Fairness Issues with Concurrency

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg P(a)}{\neg Q(b)}}{\neg S(c)}}{\forall x.\,(P(x) \vee Q(x)) \wedge \forall y.\, S(x)}}{P(X) \vee Q(X), \forall y.\, S(X)}\,\gamma_\forall}{P(X), \forall y.\, S(X)}\,\beta_\Leftrightarrow \qquad \dfrac{Q(X), \forall y.\, S(X)}{\odot}\,\beta_\Leftrightarrow}{\odot}\,\odot_\sigma \qquad \dfrac{}{\odot}\,\odot_\sigma$$

$$\sigma = \{X \mapsto a\} \qquad \sigma = \{X \mapsto b\}$$

Solving Fairness Issues with Concurrency

$$\neg P(a)$$
$$\neg Q(b)$$
$$\neg S(c)$$

$$\dfrac{\dfrac{\forall x.\ (P(x) \vee Q(x)) \wedge \forall y.\ S(x)}{P(X) \vee Q(X), \forall y.\ S(X)}\ \gamma_\forall}{\dfrac{P(X), \forall y.\ S(X)}{\odot}\ \odot_\sigma \qquad \dfrac{Q(X), \forall y.\ S(X)}{\odot}\ \odot_\sigma}\ \beta_\Leftrightarrow$$

$$\sigma = \{X \mapsto a\} \qquad \sigma = \{X \mapsto b\}$$

Solving Fairness Issues with Concurrency

$$\dfrac{\dfrac{\dfrac{\neg P(a) \quad \neg Q(b) \quad \neg S(c) \quad \forall x.\ (P(x) \vee Q(x)) \wedge \forall y.\ S(x)}{P(X) \vee Q(X), \forall y.\ S(X)}\,\gamma_\forall}{P(a), \forall y.\ S(a) \qquad Q(a), \forall y.\ S(a)}\,\beta_\Leftrightarrow}{\sigma = \{X \mapsto a\} \qquad \sigma = \{X \mapsto a\}}$$

Solving Fairness Issues with Concurrency

$$\dfrac{\dfrac{\dfrac{\neg P(a)}{\neg Q(b)}}{\neg S(c)}}{\dfrac{\dfrac{\forall x.\,(P(x) \vee Q(x)) \wedge \forall y.\,S(x)}{P(X) \vee Q(X), \forall y.\,S(X)}\,\gamma_\forall}{\dfrac{P(a), \forall y.\,S(a)}{\odot}\,\odot_\sigma \qquad \dfrac{Q(a), \forall y.\,S(a)}{}}\,\beta_\Leftrightarrow}$$

Closed

Solving Fairness Issues with Concurrency

$$\neg P(a)$$
$$\neg Q(b)$$
$$\neg S(c)$$
$$\dfrac{\dfrac{\forall x.\ (P(x) \lor Q(x)) \land \forall y.\ S(x)}{P(X) \lor Q(X), \forall y.\ S(X)}\ \gamma_\forall}{\dfrac{P(a), \forall y.\ S(a)}{\odot}\ \odot_\sigma \qquad \dfrac{Q(a), \forall y.\ S(a)}{S(a)}\ \gamma_\forall}\ \beta_\Leftrightarrow$$

Solving Fairness Issues with Concurrency

$$
\begin{array}{c}
\neg P(a) \\
\neg Q(b) \\
\neg S(c) \\
\dfrac{\dfrac{\forall x.\ (P(x) \lor Q(x)) \land \forall y.\ S(x)}{P(X) \lor Q(X), \forall y.\ S(X)}\ \gamma_\forall}{\dfrac{P(a), \forall y.\ S(a)}{\odot}\ \odot_\sigma \quad \dfrac{\dfrac{Q(a), \forall y.\ S(a)}{S(a)}\ \gamma_\forall}{\ldots}}\ \beta_\Leftrightarrow
\end{array}
$$

Open

Solving Fairness Issues with Concurrency

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\neg P(a)\\ \neg Q(b)\\ \neg S(c)\\ \forall x.\,(P(x)\vee Q(x))\wedge\forall y.\,S(x)\end{array}}{P(X)\vee Q(X),\forall y.\,S(X)}\,\gamma_{\forall}}{P(b),\forall y.\,S(b)}\,\beta_{\Leftrightarrow} \qquad \dfrac{}{Q(b),\forall y.\,S(b)}\,\beta_{\Leftrightarrow}}{}$$

$\sigma = \{X \mapsto b\}$ $\sigma = \{X \mapsto b\}$

Solving Fairness Issues with Concurrency

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg P(a)}{\neg Q(b)}}{\neg S(c)}}{\forall x.\,(P(x) \vee Q(x)) \wedge \forall y.\, S(x)}}{P(X) \vee Q(X), \forall y.\, S(X)}\,\gamma_\forall}{P(b), \forall y.\, S(b)}\,\beta_\Leftrightarrow \qquad \dfrac{Q(b), \forall y.\, S(b)}{\odot}\,\odot_\sigma$$

Closed

Solving Fairness Issues with Concurrency

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg P(a)}{\neg Q(b)}}{\neg S(c)}}{\forall x.\ (P(x) \vee Q(x)) \wedge \forall y.\ S(x)}}{P(X) \vee Q(X), \forall y.\ S(X)}\ \gamma_\forall}{P(b), \forall y.\ S(b)}\ \beta_\Leftrightarrow}{S(b)}\ \gamma_\forall \qquad \dfrac{Q(b), \forall y.\ S(b)}{\odot}\ \odot_\sigma$$

Solving Fairness Issues with Concurrency

$$\neg P(a)$$

$$\neg Q(a)$$

$$\neg S(c)$$

$$\dfrac{\forall x.\ (P(x) \lor Q(x)) \land \forall y.\ S(x)}{P(X) \lor Q(X), \forall y.\ S(X)}\ \gamma_\forall$$

$$\dfrac{P(X) \lor Q(X), \forall y.\ S(X)}{P(b), \forall y.\ S(b) \qquad Q(b), \forall y.\ S(b)}\ \beta_\Leftrightarrow$$

$$\dfrac{P(b), \forall y.\ S(b)}{\dfrac{S(b)}{\dots}}\ \gamma_\forall \qquad \dfrac{Q(b), \forall y.\ S(b)}{\odot}\ \odot_\sigma$$

Open

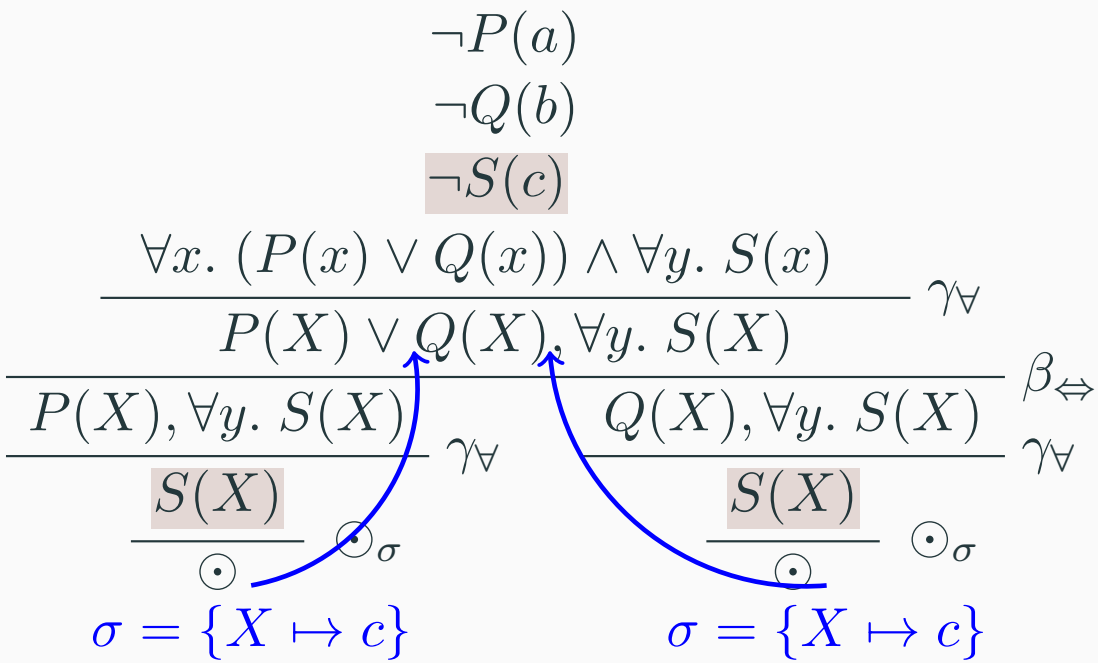
Solving Fairness Issues with Concurrency

$$\dfrac{\dfrac{\dfrac{\neg P(a)\quad \neg Q(b)\quad \neg S(c)\quad \forall x.\,(P(x)\vee Q(x))\wedge\forall y.\,S(x)}{P(X)\vee Q(X),\forall y.\,S(X)}\,\gamma_\forall}{P(X),\forall y.\,S(X)\qquad Q(X),\forall y.\,S(X)}\,\beta_\Leftrightarrow}{X\notin\{a,b\}\qquad X\notin\{a,b\}}$$

Solving Fairness Issues with Concurrency

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg P(a)}{\neg Q(b)}}{\neg S(c)}}{\dfrac{\forall x.\,(P(x) \vee Q(x)) \wedge \forall y.\, S(x)}{P(X) \vee Q(X), \forall y.\, S(X)}\,\gamma_\forall}}{\dfrac{\dfrac{P(X), \forall y.\, S(X)}{S(X)}\,\gamma_\forall \qquad \dfrac{Q(X), \forall y.\, S(X)}{S(X)}\,\gamma_\forall}{}\,\beta_\Leftrightarrow}$$

Solving Fairness Issues with Concurrency



Solving Fairness Issues with Concurrency

$$\neg P(a)$$
$$\neg Q(b)$$
$$\neg S(c)$$
$$\dfrac{\dfrac{\forall x.\ (P(x) \lor Q(x)) \land \forall y.\ S(x)}{P(c) \lor Q(c), \forall y.\ S(c)}\ \gamma_\forall}{\dfrac{\dfrac{P(c), \forall y.\ S(c)}{\dfrac{S(c)}{\odot}\ \odot_\sigma}\ \gamma_\forall \qquad \dfrac{Q(c), \forall y.\ S(c)}{\dfrac{S(c)}{\odot}\ \odot_\sigma}\ \gamma_\forall}{}}\ \beta_\Leftrightarrow$$

$$\sigma = \{X \mapsto c\} \qquad \sigma = \{X \mapsto c\}$$

Contributions

- A Concurrent tableau-based proof-search procedure
- Concurrent exploration of branches
- Eager closure
- Backtrack and forbidden substitutions
- Tackle fairness challenges
- Completeness proof of the procedure
- Implemented into a tool: Goéland

3. Goéland: Implementation, Experiments, and Analysis

Goéland Tool

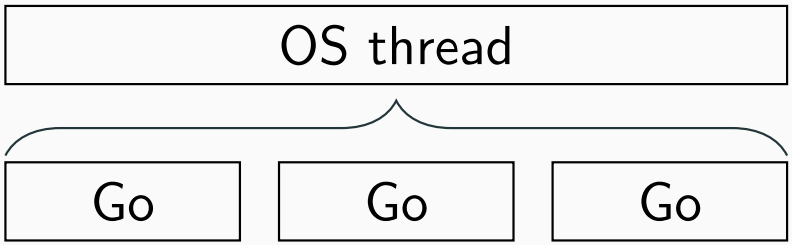
Functionnalities

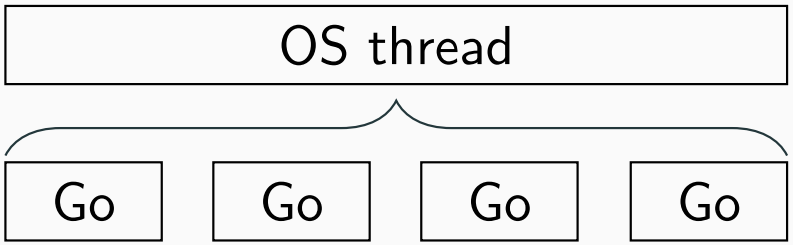
- Concurrent proof-search procedure
- Equality reasoning
- Deduction modulo theory (+ polarized)
- Polymorphic types
- Alternative modes: incomplete, interactive,...

Goéland Tool

Implementation

- 40 000 lines of code
- Go programming language
- Designed for concurrency
- Goroutines: N:M lightweight threads





Theory Reasoning

Motivation and Challenges

- Reason within specific contexts (arithmetic, industrial problems, ...)
- Deal with a large number of axioms
- Handle multiple theories

Background Reasoners

- Equality
- Deduction modulo theory (DMT)

Deduction Modulo Theory

Principle

- Turns axioms into rewrite rules
- Triggers only relevant axioms
- Produces shorter proof
- Not limited to one theory

Main Heuristic

$(\forall \vec{x}.)\ A \Leftrightarrow F$ where:

- A is an atomic formula
- F is a non-atomic formula

Polarized DMT

$(\forall \vec{x}.)\ A \Rightarrow F$ where:

- A is an atomic formula
- F is a non-atomic formula

Axiom: $\forall x.\ P(x) \Leftrightarrow \forall y.\ Q(x,y) \wedge S(x,y)$

Rule: $P(X) \rightarrow \forall y.\ Q(X,y) \wedge S(X,y)$

Protocol of the Experiments

- Thousand of Problems for Theorem Provers (TPTP) library (v8.1.2)
- Syntactic (SYN) and set theory (SET) categories
- First-order logic (FOL)
- Goéland and its variants, Zenon (+ modulo), Princess, Vampire and E
- 300 seconds of timeout
- Intel Xeon E5-2680 v4 2.4GHz 2×14-core processor with 128GB

Goéland Variants over SYN and SET

	SYN (288 problems)		SET (464 problems)	
Goéland	**209**	(1.2 s)	**124**	(18.6 s)
Goéland+EQ	**213**	(0.3 s)	**101**	(15.6 s)
Goéland+DMT	**209**	(1.3 s)	**217**	(5.9 s)
Goéland+DMT+EQ	**213**	(0.5 s)	**192**	(10.2 s)
Goéland+DMT +Polarized	**202**	(0.3 s)	**164**	(1.5 s)

All Provers on FOF

	FOF (5396 problems)
Goéland	**613** (10 482 s — 17.1 s)
Goéland+DMT	**770** (6 935 s — 9 s)
Goéland+DMT+EQ	**801** (10 060 s — 12.5 s)
Zenon	**1 382** (9 026 s — 6.5 s)
Zenon Modulo	**1 389** (10 028 s — 7.2 s)
Princess	**1 621** (23 200 s — 14.3 s)
Vampire	**3 342** (42 873 s — 12.8 s)
E	**3 939** (39 638 s — 10.1 s)

Analysis

- Promising results
- Less problems solved than other ATP
- Scaling issue
- Memory management
- Equality reasoning performances
- Good results with DMT

4. Toward Certification: an Output for Checkable Proofs

Advanced Skolemization Strategies

Motivations

- Shorter proofs
- Faster proof search

Inner Skolemization (δ^+-rule)

- Extension of δ-rule
- Uses only the free variables of the formula

Pre-Inner Skolemization (δ^{+^+}-rule)

- Extension of δ^+-rule
- Reuses the same Skolem symbol if they come from α-equivalent formulas

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\,\gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\,\alpha_{\neg\Rightarrow}}{\neg D(f(X))}\,\delta_{\neg\forall}}{\neg(D(X_2) \Rightarrow \forall y\ D(y))}\,\gamma_{\neg\exists}}{D(X_2), \neg\forall y\ D(y)}\,\alpha_{\neg\Rightarrow}}{\sigma = \{X_2 \mapsto f(X)\}}\,\odot_\sigma$$

(a) Outer Skolemization tableau.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\,\gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\,\alpha_{\neg\Rightarrow}}{\neg D(c)}\,\delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\,\odot_\sigma$$

(b) Inner Skolemization tableau.

Translation to Machine-Checkable Proofs

Gentzen-Schütte Calculus (GS3)

- Equivalent to tableaux
- Easily translatable to proof assistants
- Only supports outer skolemization

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\, D(x) \Rightarrow \forall y\, D(y))}{\neg(D(X) \Rightarrow \forall y\, D(y))}\,\gamma_{\neg\exists}}{D(X), \neg(\forall y\, D(y))}\,\alpha_{\neg\Rightarrow}}{\neg D(f(X))}\,\delta_{\neg\forall}}{\neg(D(X_2) \Rightarrow \forall y\, D(y))}\,\gamma_{\neg\exists}}{D(X_2), \neg\forall y\, D(y)}\,\alpha_{\neg\Rightarrow}}{\sigma = \{X_2 \mapsto f(X)\}}\,\odot_{\sigma}$$

(a) Outer Skolemization tableau proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\ldots, \neg D(c'), D(c'), \neg(\forall y\, D(y)) \vdash}\,\text{ax}}{\ldots, \neg(D(c') \Rightarrow \forall y\, D(y)) \vdash}\,\neg\Rightarrow}{\neg(\exists x.\, D(x) \Rightarrow \forall y\, D(y)), \ldots, \neg D(c') \vdash}\,\neg\exists}{\ldots, D(c), \neg(\forall y\, D(y)) \vdash}\,\neg\forall}{\ldots, \neg(D(c) \Rightarrow \forall y\, D(y)) \vdash}\,\neg\Rightarrow}{\neg(\exists x.\, D(x) \Rightarrow \forall y\, D(y)) \vdash}\,\neg\exists$$

(b) Equivalent GS3.

Outer Skolemisation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\,\gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\,\alpha_{\neg\Rightarrow}}{\neg D(c)}\,\delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\,\odot_{\sigma}$$

(a) Inner Skolemization tableau proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\ldots, D(c), \neg(\forall y\ D(y)), \neg D(c) \vdash}\,\text{ax}}{\ldots, D(c), \neg(\forall y\ D(y)) \vdash}\,\neg_{\forall}\ (\star)}{\ldots, \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\,\neg_{\Rightarrow}}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)) \vdash}\,\neg_{\exists}$$

(b) Incorrect equivalent GS3.

A Deskolemization Strategy

Idea

Perform all the Skolemization steps before the other rules, so the Skolem symbol is necessarily fresh.

Key Notions

- Formulas that depend on a Skolem symbol
- Formulas that descend from a Skolem symbol
- A formula F needs to be processed before another formula G iff G makes use of a Skolem symbol generated by F

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\,\gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\,\alpha_{\neg\Rightarrow}}{\neg D(c)}\,\delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\,\odot_{\sigma}$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\,\gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\,\alpha_{\neg\Rightarrow}}{\neg D(c)}\,\delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\,\odot_{\sigma}$$

$$\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)) \vdash$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\,\gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\,\alpha_{\neg\Rightarrow}}{\neg D(c)}\,\delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\,\odot_{\sigma}$$

$$\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y))}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\,\neg\exists$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\,\gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\,\alpha_{\neg\Rightarrow}}{\neg D(c)}\,\delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\,\odot_{\sigma}$$

$$\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y))}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\,\neg\exists$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\,\gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\,\alpha_{\neg\Rightarrow}}{\neg D(c)}\,\delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\,\odot_{\sigma}$$

$$\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\,\neg\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\,\neg\exists$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\,\gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\,\alpha_{\neg\Rightarrow}}{\neg D(c)}\,\delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\,\odot_{\sigma}$$

$$\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\,\neg\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\,\neg\exists$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\,\gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\,\alpha_{\neg\Rightarrow}}{\neg D(c)}\,\delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\,\odot_{\sigma}$$

$$\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\,\neg\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\,\neg\exists$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\,\gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\,\alpha_{\neg\Rightarrow}}{\neg D(c)}\,\delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\,\odot_{\sigma}$$

$$\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\,\mathsf{W} \times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\,\neg\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\,\neg\exists$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\,\gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\,\alpha_{\neg\Rightarrow}}{\neg D(c)}\,\delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\,\odot_{\sigma}$$

$$\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\,\mathsf{W} \times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\,\neg\Rightarrow$$
$$\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\,\neg\exists$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\,\gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\,\alpha_{\neg\Rightarrow}}{\neg D(c)}\,\delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\,\odot_{\sigma}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}\,\neg\forall}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\,\mathsf{W}\times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\,\neg\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\,\neg\exists$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\,\gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\,\alpha_{\neg\Rightarrow}}{\neg D(c)}\,\delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\,\odot_{\sigma}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}\,\neg\forall}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\,\mathsf{W}\times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\,\neg\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\,\neg\exists$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\,\gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\,\alpha_{\neg\Rightarrow}}{\neg D(c)}\,\delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\,\odot_{\sigma}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}\,\neg\forall}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\,\mathsf{W}\times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\,\neg\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\,\neg\exists$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\,\gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\,\alpha_{\neg\Rightarrow}}{\neg D(c)}\,\delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\,\odot_{\sigma}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}\,\neg\exists}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}\,\neg\forall}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\,\mathsf{W} \times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\,\neg\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\,\neg\exists$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\,\gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\,\alpha_{\neg\Rightarrow}}{\neg D(c)}\,\delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\,\odot_{\sigma}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}\,\neg\exists}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}\,\neg\forall}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\,\mathsf{W} \times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\,\neg\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\,\neg\exists$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\,\gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\,\alpha_{\neg\Rightarrow}}{\neg D(c)}\,\delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\,\odot_{\sigma}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}\,\neg\exists}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}\,\neg\forall}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\,\mathsf{W} \times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\,\neg\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\,\neg\exists$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\,\gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\,\alpha_{\neg\Rightarrow}}{\neg D(c)}\,\delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\,\odot_{\sigma}$$

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Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\,\gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\,\alpha_{\neg\Rightarrow}}{\neg D(c)}\,\delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\,\odot_{\sigma}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)), \neg D(c) \vdash}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}\,\neg\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)), \neg D(c) \vdash}\,\neg\exists}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(\forall y\ D(y)) \vdash}\,\neg\forall}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)), D(c), \neg(\forall y\ D(y)) \vdash}\,\mathsf{W} \times 2}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y)), \neg(D(c) \Rightarrow \forall y\ D(y)) \vdash}\,\neg\Rightarrow}{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}\,\neg\exists$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\exists x.\ D(x) \Rightarrow \forall y\ D(y))}{\neg(D(X) \Rightarrow \forall y\ D(y))}\,\gamma_{\neg\exists}}{D(X), \neg(\forall y\ D(y))}\,\alpha_{\neg\Rightarrow}}{\neg D(c)}\,\delta^{+}_{\neg\forall}}{\sigma = \{X \mapsto c\}}\,\odot_{\sigma}$$

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Implementation

Implementation

- δ, δ^+ and δ^{+^+} Skolemization strategies
- GS3 proofs
- Deskolemization algorithms
- Coq & Lambdapi outputs

Evaluation Protocol

- Same setup as previous tests
- 3 Skolemization strategies + DMT
- Number of problems solved
- Size of the proof (number of branches)

Results

	Problems Proved	Percentage Certified	Avg. Size Increase	Max. Size Increase
Goéland	261	100 %	0 %	-
Goéland+δ^+	272	100 %	8.1 %	5.3
Goéland+δ^{+^+}	274	100 %	10.6 %	10.3
Goéland+DMT	363	100 %	0 %	-
Goéland+DMT+δ^+	375	100 %	4.5 %	3.9
Goéland+DMT+δ^{+^+}	377	100 %	7.4 %	5.2

Contributions

- An optimization of the deskolemization algorithm for δ^+
- A deskolemization algorithm for δ^{+^+}
- Soundness proof for both translations
- Output of GS3 proof into Coq and Lambdapi
- Promising results
- 100% of the proofs are certified
- Far below the theoretical bound

Conclusion

Contributions

Fairness in Tableau-Based Proof Search

- Fairness between branches managed by concurrency
- Completeness of the procedure

Theory Reasoning in Tableaux

- Implementation of two background reasoners
- Study of parallelization points and interaction with the proof search

Proof Certification

- A sound generic deskolemization algorithm
- Output of the proofs into Coq & Lambdapi

Perspectives

Fairness in Tableau-Based Proof Search

- Improvement of the performances of Goéland
- Heuristics in formula computation order, closure management
- Simulate "intuition" with learning methods
- Modular and generic prover

Theory Reasoning

- Improvements of DMT (term rewriting, narrowing, manually designed rewrite rules)
- More experiments on polymorphic problems
- SMT solvers: string equations, quantum circuits

Proof Certification

- Reduce the number of branches by the use of lemmas
- Framework for verification of tableau proofs

Thank you! ☺

First-Order Logic

Conventions

- Constant symbols: a, b, c
- Function symbols: f, f'
- Bound and free variables: x, y, X, X_2, Y
- Predicate symbols: P, Q, S
- Connectives: $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$
- Quantifiers: $\exists, \forall$

- $Socrates$
- $Human(x)$
- $Mortal(x)$

- All humans are mortals
- Socrates is a human
- Then Socrates is mortal

$$(\forall x.\ Human(x) \Rightarrow Mortal(x)) \wedge Human(Socrates) \Rightarrow Mortal(Socrates)$$

Reasoning Methods in FOL

Resolution

- Breaks the initial formula into clauses
- Derivation step and saturation
- More efficient

CNF : $\{\neg A\}, \{A \vee A\}, \{\neg B \vee A\}$

$\{\neg A\}$

$\{A \vee A\}$

$\{\neg B \vee A\}$

Tableaux

- Works with the unaltered original formula
- Reduces the goal into subgoals
- Better interoperability

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)}{(A \Rightarrow B) \Rightarrow A, \neg A}\,\alpha_{\neg\Rightarrow}}{\neg(A \Rightarrow B)}\,\beta_{\Rightarrow} \qquad \dfrac{A}{\odot}\,\odot}{A, \neg B}\,\alpha_{\neg\Rightarrow}}{\odot}\,\odot$$

Reasoning Methods in FOL

Resolution

- Breaks the initial formula into clauses
- Derivation step and saturation
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CNF : $\{\neg A\}, \{A \vee A\}, \{\neg B \vee A\}$

$\{\neg A\}$

$\{A \vee A\}$ —— $\{A\}$

$\{\neg B \vee A\}$

Tableaux

- Works with the unaltered original formula
- Reduces the goal into subgoals
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$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)}{(A \Rightarrow B) \Rightarrow A, \neg A}\,\alpha_{\neg\Rightarrow}}{\neg(A \Rightarrow B)}\,\beta_{\Rightarrow} \qquad \dfrac{A}{\odot}\,\odot}{A, \neg B}\,\alpha_{\neg\Rightarrow}}{\odot}\,\odot$$

Reasoning Methods in FOL

Resolution

- Breaks the initial formula into clauses
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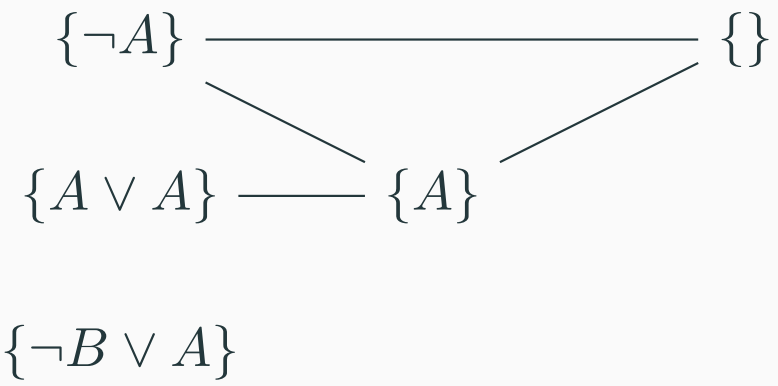
CNF : $\{\neg A\}, \{A \vee A\}, \{\neg B \vee A\}$



Tableaux

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$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)}{(A \Rightarrow B) \Rightarrow A, \neg A}\,\alpha_{\neg\Rightarrow}}{\neg(A \Rightarrow B)}\,\beta_{\Rightarrow} \qquad \dfrac{A}{\odot}\,\odot}{A, \neg B}\,\alpha_{\neg\Rightarrow}}{\odot}\,\odot$$

Select Branch

$$\dfrac{\dfrac{\dfrac{(P(a) \wedge \neg P(a)) \vee \bot}{P(a) \wedge P(a)}\,\alpha_\wedge \qquad \dfrac{\bot}{\odot}\,\odot}{P(a), \neg P(a)}\,\odot}{\odot}\,\beta_\vee$$

(a) Proof of $(P(a) \wedge \neg P(a)) \vee \bot$.

$$\dfrac{\dfrac{\dfrac{(P(a) \wedge \neg P(a)) \vee \bot}{P(a) \wedge P(a)}\,\alpha_\wedge \qquad \dfrac{\bot}{\odot}\,\odot}{P(a), \neg P(a)}\,\odot}{\odot}\,\beta_\vee \ldots$$

(b) Incompleteness caused by an unfair *select branch*.

Select Formula

$$\dfrac{\dfrac{\dfrac{P(a) \wedge \neg P(a)}{\forall x\, Q(X)}}{P(a), \neg P(a)}\,\alpha_\wedge}{\odot}\,\odot$$

(a) Proof of $P(a) \wedge \neg P(a), \forall x\, Q(X)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{P(a) \wedge \neg P(a)}{\forall x\, Q(X)}\,\gamma_\forall}{Q(X)}\,\gamma_\forall}{Q(X')}\,\gamma_\forall}{\cdots}$$

(b) Incompleteness caused by an unfair *select formula*.

Select Pair (Right Branch First)

$$P(a)$$
$$\neg P(b)$$
$$\forall x.\ P(x) \Leftrightarrow (\forall y\ P(y))$$

Select Pair (Right Branch First)

$$\dfrac{\begin{array}{c} P(a) \\ \neg P(b) \\ \forall x.\ P(x) \Leftrightarrow (\forall y\ P(y)) \end{array}}{P(X) \Leftrightarrow (\forall y\ P(y))}\ \gamma_{\forall}$$

Select Pair (Right Branch First)

$$\dfrac{\dfrac{\begin{array}{c}P(a)\\ \neg P(b)\\ \forall x.\ P(x) \Leftrightarrow (\forall y\ P(y))\end{array}}{P(X) \Leftrightarrow (\forall y\ P(y))}\,\gamma_\forall}{P(X), \forall y\ P(y) \qquad \neg P(X), \neg(\forall y\ P(y))}\,\beta_\Leftrightarrow$$

Select Pair (Right Branch First)

$$\dfrac{\dfrac{\dfrac{P(a)}{\neg P(b)}\qquad\dfrac{\forall x.\ P(x) \Leftrightarrow (\forall y\ P(y))}{P(X) \Leftrightarrow (\forall y\ P(y))}\,\gamma_\forall}{P(X), \forall y\ P(y) \qquad \dfrac{\neg P(X), \neg(\forall y\ P(y))}{\sigma = \{X \mapsto a\}}\,\odot_\sigma}\,\beta_\Leftrightarrow}$$

Select Pair (Right Branch First)

$$\dfrac{\dfrac{\begin{array}{c}P(a)\\ \neg P(b)\\ \dfrac{\forall x.\ P(x) \Leftrightarrow (\forall y\ P(y))}{P(a) \Leftrightarrow (\forall y\ P(y))}\,\gamma_\forall\end{array}}{P(a), \forall y\ P(y) \qquad \dfrac{\neg P(a), \neg(\forall y\ P(y))}{\sigma = \{X \mapsto a\}}\,\odot_\sigma}\,\beta_\Leftrightarrow$$

Select Pair (Right Branch First)

$$\dfrac{\dfrac{P(a) \quad \neg P(b) \quad \dfrac{\forall x.\, P(x) \Leftrightarrow (\forall y\ P(y))}{P(a) \Leftrightarrow (\forall y\ P(y))}\,\gamma_\forall}{\dfrac{P(a), \forall y\ P(y)}{P(Y)}\,\gamma_\forall \qquad \dfrac{\neg P(a), \neg(\forall y\ P(y))}{\sigma = \{X \mapsto a\}}\,\odot_\sigma}\,\beta_\Leftrightarrow$$

Select Pair (Right Branch First)

$$\begin{array}{c} P(a) \\ \neg P(b) \\ \dfrac{\dfrac{\forall x.\ P(x) \Leftrightarrow (\forall y\ P(y))}{P(a) \Leftrightarrow (\forall y\ P(y))}\,\gamma_\forall}{\dfrac{\dfrac{P(a), \forall y\ P(y)}{P(Y)}\,\gamma_\forall}{\sigma = \{Y \mapsto b\}}\,\odot_\sigma \qquad \dfrac{\neg P(a), \neg(\forall y\ P(y))}{\sigma = \{X \mapsto a\}}\,\odot_\sigma}\,\beta_\Leftrightarrow \end{array}$$

Select Pair (Right Branch First)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(a)}{\neg P(b)}\quad \dfrac{\forall x.\ P(x) \Leftrightarrow (\forall y\ P(y))}{P(a) \Leftrightarrow (\forall y\ P(y))}\,\gamma_\forall}{\dfrac{P(a), \forall y\ P(y)}{P(b)}\,\gamma_\forall}}{\sigma = \{Y \mapsto b\}}\,\odot_\sigma \qquad \dfrac{\neg P(a), \neg(\forall y\ P(y))}{\sigma = \{X \mapsto a\}}\,\odot_\sigma}{}\,\beta_\Leftrightarrow$$

Select Pair (Left Branch First)

$$P(a)$$
$$\neg P(b)$$
$$\forall x.\ P(x) \Leftrightarrow (\forall y\ P(y))$$

Select Pair (Left Branch First)

$$\frac{\begin{array}{c} P(a) \\ \neg P(b) \\ \forall x.\, P(x) \Leftrightarrow (\forall y\ P(y)) \end{array}}{P(X) \Leftrightarrow (\forall y\ P(y))}\ \gamma_{\forall}$$

Select Pair (Left Branch First)

$$\dfrac{\dfrac{\begin{array}{c}P(a)\\ \neg P(b)\\ \forall x.\ P(x) \Leftrightarrow (\forall y\ P(y))\end{array}}{P(X) \Leftrightarrow (\forall y\ P(y))}\,\gamma_{\forall}}{P(X), \forall y\ P(y) \qquad\qquad \neg P(X), \neg(\forall y\ P(y))}\,\beta_{\Leftrightarrow}$$

Select Pair (Left Branch First)

$$\dfrac{\dfrac{\dfrac{P(a)}{\neg P(b)}\quad\dfrac{\forall x.\ P(x) \Leftrightarrow (\forall y\ P(y))}{P(X) \Leftrightarrow (\forall y\ P(y))}\,\gamma_\forall}{\dfrac{P(X), \forall y\ P(y)}{\sigma = \{X \mapsto b\}}\,\odot_\sigma \qquad \neg P(X), \neg(\forall y\ P(y))}\,\beta_\Leftrightarrow$$

Select Pair (Left Branch First)

$$\dfrac{\dfrac{\dfrac{P(a) \quad \neg P(b) \quad \dfrac{\forall x.\ P(x) \Leftrightarrow (\forall y\ P(y))}{P(b) \Leftrightarrow (\forall y\ P(y))}\,\gamma_\forall}{P(b), \forall y\ P(y)}\,\beta_\Leftrightarrow \qquad \neg P(b), \neg(\forall y\ P(y))}{\sigma = \{X \mapsto b\}}\,\odot_\sigma$$

Select Pair (Left Branch First)

$$\dfrac{\dfrac{P(a) \quad \neg P(b) \quad \dfrac{\forall x.\ P(x) \Leftrightarrow (\forall y\ P(y))}{P(b) \Leftrightarrow (\forall y\ P(y))}\,\gamma_{\forall}}{\dfrac{P(b), \forall y\ P(y)}{\sigma = \{X \mapsto b\}}\,\odot_{\sigma} \qquad \dfrac{\neg P(b), \neg(\forall y\ P(y))}{\neg P(f(b))}\,\delta_{\neg\forall}}\,\beta_{\Leftrightarrow}}$$

Select Pair (Left Branch First)

$$\dfrac{\dfrac{\dfrac{P(a) \quad \neg P(b) \quad \forall x.\, P(x) \Leftrightarrow (\forall y\ P(y))}{P(b) \Leftrightarrow (\forall y\ P(y))}\,\gamma_\forall}{\dfrac{P(b), \forall y\ P(y)}{\sigma = \{X \mapsto b\}}\,\odot_\sigma \qquad \dfrac{\dfrac{\neg P(b), \neg(\forall y\ P(y))}{\neg P(f(b))}\,\delta_{\neg\forall}}{P(X_2) \Leftrightarrow (\forall y\ P(y))}\,\gamma_\forall}\,\beta_\Leftrightarrow$$

Select Pair (Left Branch First)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(a)}{\neg P(b)}\quad \dfrac{\forall x.\ P(x) \Leftrightarrow (\forall y\ P(y))}{P(b) \Leftrightarrow (\forall y\ P(y))}\,\gamma_\forall}{\dfrac{P(b), \forall y\ P(y)}{\sigma = \{X \mapsto b\}}\,\odot_\sigma}\,\beta_\Leftrightarrow \qquad \dfrac{\dfrac{\dfrac{\neg P(b), \neg(\forall y\ P(y))}{\neg P(f(b))}\,\delta_{\neg\forall}}{P(X_2) \Leftrightarrow (\forall y\ P(y))}\,\gamma_\forall}{P(X_2), \forall y\ P(y) \qquad \neg P(X_2), \neg(\forall y\ P(y))}\,\beta_\Leftrightarrow$$

Select Pair (Left Branch First)

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\qquad
\dfrac{\dfrac{\dfrac{\neg P(b), \neg(\forall y\ P(y))}{\neg P(f(b))}\ \delta_{\neg\forall}}{P(X_2) \Leftrightarrow (\forall y\ P(y))}\ \gamma_\forall}{\neg P(X_2), \neg(\forall y\ P(y))}\ \beta_\Leftrightarrow
}{P(X_2), \forall y\ P(y)}\ \odot_\sigma
}{\sigma = \{X_2 \mapsto b\}}
}{\sigma' = \{X_2 \mapsto f(b)\}}
$$

Select Pair (Left Branch First)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(a)}{\neg P(b)}}{\dfrac{\forall x.\ P(x) \Leftrightarrow (\forall y\ P(y))}{P(b) \Leftrightarrow (\forall y\ P(y))}\,\gamma_\forall}}{\dfrac{P(b), \forall y\ P(y)}{\sigma = \{X \mapsto b\}}\,\odot_\sigma} \qquad \dfrac{\dfrac{\dfrac{\neg P(b), \neg(\forall y\ P(y))}{\neg P(f(b))}\,\delta_{\neg\forall}}{P(b) \Leftrightarrow (\forall y\ P(y))}\,\gamma_\forall}{\dfrac{P(b), \forall y\ P(y)}{\sigma = \{X_2 \mapsto b\}}\,\odot_\sigma} \qquad \neg P(b), \neg(\forall y\ P(y))}{\sigma' = \{X_2 \mapsto f(b)\}}\,\beta_\Leftrightarrow}{}\,\beta_\Leftrightarrow$$

Select Pair (Left Branch First)

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}{\sigma = \{X \mapsto b\}}\ \odot_{\sigma}
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\neg P(b), \neg(\forall y\ P(y))
}{\neg P(f(b))}\ \delta_{\neg\forall}
}{P(b) \Leftrightarrow (\forall y\ P(y))}\ \gamma_{\forall}
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}{\sigma = \{X_2 \mapsto b\}}\ \odot_{\sigma}
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\neg P(b), \neg(\forall y\ P(y))
}{\neg P(f'(b))}\ \delta_{\neg\forall}
}{\sigma' = \{X_2 \mapsto f(b)\}}
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Select Pair (Left Branch First)

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P(a) \quad \neg P(b) \quad \forall x.\, P(x) \Leftrightarrow (\forall y\ P(y))
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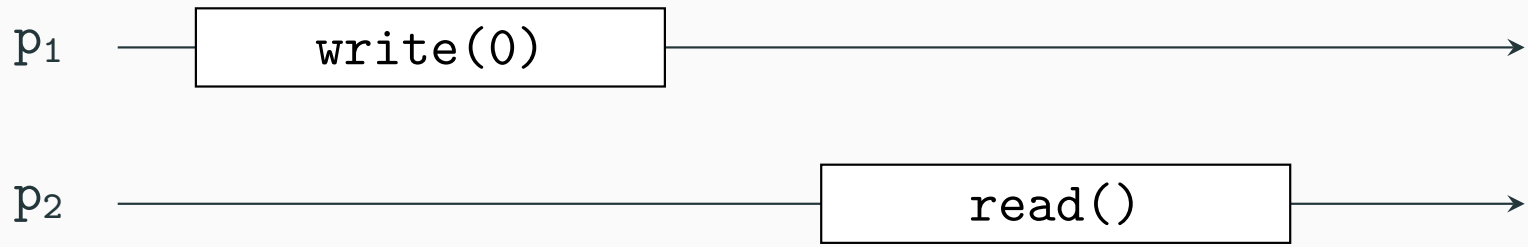
}{\neg P(b), \neg(\forall y\ P(y))}\,\beta_\Leftrightarrow
}{\neg P(f(b))}\,\delta_{\neg\forall}
}{P(b) \Leftrightarrow (\forall y\ P(y))}\,\gamma_\forall
}{\neg P(b), \neg(\forall y\ P(y))}\,\beta_\Leftrightarrow
}{\neg P(f'(b))}\,\delta_{\neg\forall}
}{\cdots}\,\gamma_\forall
}{\sigma = \{X_2 \mapsto b\}}\,\odot_\sigma
}{P(b), \forall y\ P(y)}\,\beta_\Leftrightarrow
}{\sigma = \{X \mapsto b\}}\,\odot_\sigma
}{P(b), \forall y\ P(y)}\,\beta_\Leftrightarrow
$$

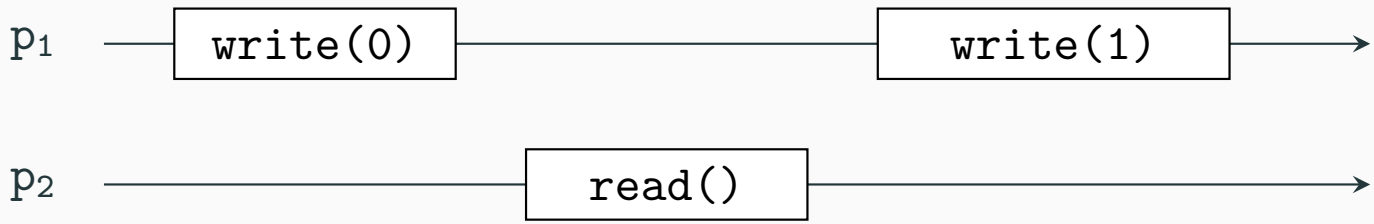
Sequential and Concurrent Executions

(a) Sequential execution of two operations on a resource `R` by the process p_1.

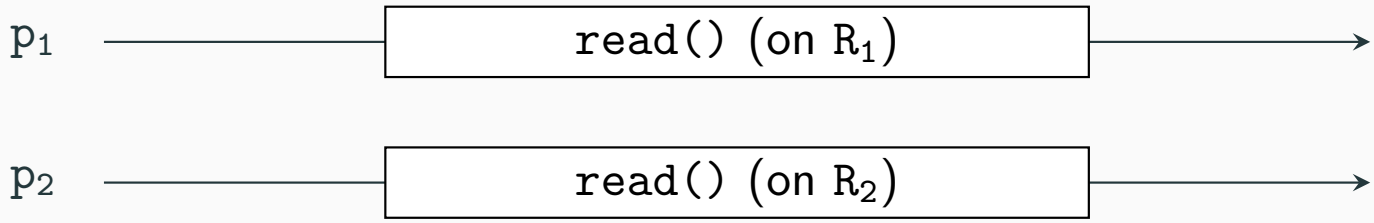


(b) Concurrent execution of two operations on a shared resource `R`: `write(0)` by p_1 and `read()` by p_2.

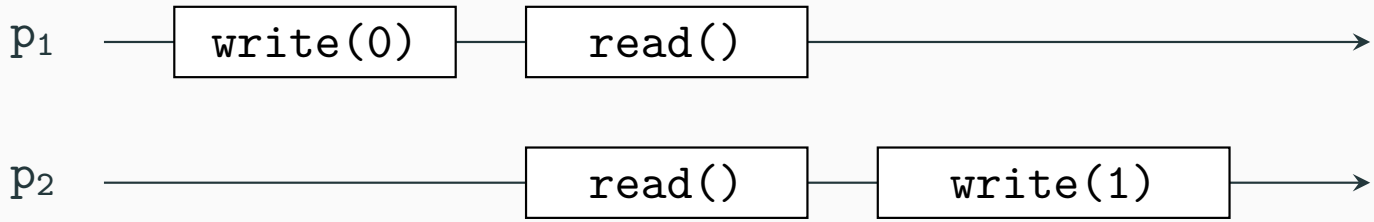
Parallelism and Concurrency



(a) *Concurrent* but not *parallel*.



(b) *Parallel* but not *concurrent*.



(c) *Concurrent* and *parallel*.

Interactions with the Proof-Search Procedure

Equality Reasoning

- Capture equality predicate $\approx$
- Extract terms that have to be equals
- $\mathcal{BSE}$ calculus
- New closure rule: $\neg(X \approx X)$

Integration

- Triggered when a predicate or an equality is generated
- Backtrack if the chosen solution does not fit with other branches
- Parallelization if multiple rules are applicable

$\mathcal{BSE}$-Calculus

Rigid E-unification Problem

A rigid E-unification problem

$$\langle E, s, t \rangle$$

consists of a finite set E of equalities of the form $(l \approx r)$ and two terms s and t such that $r, l, s, t \in \mathscr{T}$.

Constraint

An (ordering) constraint is a (finite) set of expressions of the form $s \simeq t$ or $s \succ t$ where s and t are terms. A substitution σ is a solution to a constraint $\mathcal{C}$ if and only if :

- $\sigma(s) = \sigma(t)$ for all $s \simeq t \in \mathcal{C}$, i.e., σ is an unifier for s and t.
- $\sigma(s) > \sigma(t)$ for all $s \succ t \in \mathcal{C}$, where $>$ is an arbitrary but fixed term reduction ordering.
- σ instantiates all variables occurring in $\mathcal{C}$ with ground terms.

Basic Rigid Superposition Rules

Right Basic Rigid Superposition Rule

Let $l \approx r$ or $r \approx l$ be an equality of E and l' is sub-term of s or t. Thus, the application of the right rigid basic superposition (*rbrs*) rule results in one of the following, regarding the term on which it is applied:

- s become $s_{[l' \mapsto r]}$ and the constraints $l \succ r$, $s \succ t$, $l \simeq l'$ are added to C
- t become $t_{[l' \mapsto r]}$ and the constraints $l \succ r$, $t \succ s$, $l \simeq l'$ are added to C

$$\frac{\langle E \cup \{l \approx r\}, s, t\rangle \cdot \mathcal{C}}{\langle E \cup \{l \approx r\}, s_{[l' \mapsto r]}, t\rangle \cdot \mathcal{C} \cup \{l \succ r, s \succ t, l \simeq l'\}}\ rbrs$$

Basic Rigid Superposition Rules

Left Basic Rigid Superposition Rule

Let $l \approx r$ or $r \approx l$ and $u \approx v$ or $v \approx u$ be two equalities of E. Let l' be a sub-term of u. Thus, the application of the left rigid basic superposition (*lbrs*) rule results in $u \approx v$ becoming $u_{[l' \mapsto r]} \approx v$ and the constraints $l \succ r$, $u \succ v$ and $l \simeq l'$ are added to C.

$$\frac{\langle E \cup \{l \approx r, u \approx v\}, s, t\rangle \cdot \mathcal{C}}{\langle E \cup \{l \approx r, u_{[l' \mapsto r]} \approx v\}, s, t\rangle \cdot \mathcal{C} \cup \{l \succ r, u \succ v, l \simeq l'\}}\ lbrs$$

Example using Equality Reasoning

$$
\begin{gathered}
a \approx b \\
a \approx c \\
\neg P(c, c) \\
\forall x.\ P(c, c) \vee \neg (x \approx c)
\end{gathered}
$$

Example using Equality Reasoning

$$\frac{\begin{array}{c} a \approx b \\ a \approx c \\ \neg P(a,b) \\ \boldsymbol{\forall x.\ P(c,c) \vee \neg(x \approx c)} \end{array}}{P(c,c) \vee \neg(X \approx c)}\ \gamma_{\forall}$$

Example using Equality Reasoning

$$\dfrac{\dfrac{\dfrac{\begin{array}{c} a \approx b \\ a \approx c \\ \neg P(c,c) \\ \forall x.\ P(c,c) \vee \neg(x \approx c) \end{array}}{\boldsymbol{P(c,c) \vee \neg(X \approx c)}}\,\gamma_\forall}{P(c,c) \qquad \neg(X \approx c)}\,\beta_\vee$$

Example using Equality Reasoning

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c} a \approx b \\ a \approx c \\ \neg \boldsymbol{P(a,b)} \\ \forall x.\ P(c,c) \vee \neg(x \approx c) \end{array}}{P(c,c) \vee \neg(X \approx c)}\,\gamma_{\forall}}{\boldsymbol{P(c,c)} \qquad \neg(X \approx c)}\,\beta_{\vee}
$$

Example using Equality Reasoning

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c} a \approx b \\ a \approx c \\ \neg \boldsymbol{P(a,b)} \\ \forall x.\, P(c,c) \vee \neg(x \approx c) \end{array}}{P(c,c) \vee \neg(X \approx c)}\,\gamma_{\forall}}{\boldsymbol{P(c,c)} \qquad \neg(X \approx c)}\,\beta_{\vee}
$$

$$\langle \{a \approx b, a \approx c\}, a, c \rangle \cdot \emptyset$$

Example using Equality Reasoning

$$\dfrac{\dfrac{\dfrac{\begin{array}{c} a \approx b \\ a \approx c \\ \neg \boldsymbol{P(a,b)} \end{array}}{\forall x.\, P(c,c) \vee \neg(x \approx c)}\,\gamma_{\forall}}{P(c,c) \vee \neg(X \approx c)}\,\beta_{\vee}}{\boldsymbol{P(c,c)} \qquad \neg(X \approx c)}$$

$$\dfrac{\langle \{a \approx b, a \approx c\}, a, c\rangle \cdot \emptyset}{\langle \{a \approx b, a \approx c\}, c, c\rangle \cdot \{a \succ c, a \simeq a\}}\; rbrs$$

Example using Equality Reasoning

$$\dfrac{\dfrac{\dfrac{\begin{array}{c} a \approx b \\ a \approx c \\ \neg \boldsymbol{P(a,b)} \end{array}}{\forall x.\, P(c,c) \vee \neg(x \approx c)}\,\gamma_{\forall}}{P(c,c) \vee \neg(X \approx c)}\,\beta_{\vee}}{\boldsymbol{P(c,c)} \qquad \neg(X \approx c)}$$

$$\dfrac{\langle \{a \approx b, a \approx c\}, a, c\rangle \cdot \emptyset}{\langle \{a \approx b, a \approx c\}, c, c\rangle \cdot \{a \succ c, a \simeq a\}}\; rbrs$$

Example using Equality Reasoning

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c} a \approx b \\ a \approx c \\ \boldsymbol{\neg P(c,b)} \\ \forall x.\, P(c,c) \vee \neg(x \approx c) \end{array}}{P(c,c) \vee \neg(X \approx c)}\,\gamma_{\forall}}{\boldsymbol{P(c,c)} \qquad \neg(X \approx c)}\,\beta_{\vee}
$$

Example using Equality Reasoning

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c} a \approx b \\ a \approx c \\ \boldsymbol{\neg P(c,b)} \\ \forall x.\, P(c,c) \vee \neg(x \approx c) \end{array}}{P(c,c) \vee \neg(X \approx c)}\,\gamma_{\forall}}{\boldsymbol{P(c,c)} \qquad \neg(X \approx c)}\,\beta_{\vee}
$$

$$
\langle \{a \approx b, a \approx c\}, b, c \rangle \cdot \emptyset
$$

Example using Equality Reasoning

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
a \approx b \qquad a \approx c \qquad \boldsymbol{\neg P(c,b)} \qquad \forall x.\, P(c,c) \vee \neg(x \approx c)
}{P(c,c) \vee \neg(X \approx c)}\,\gamma_{\forall}
}{\boldsymbol{P(c,c)} \qquad \neg(X \approx c)}\,\beta_{\vee}
}{\langle \{a \approx b, a \approx c\}, b, c\rangle \cdot \emptyset}
}{\langle \{a \approx b, a \approx c\}, a, c\rangle \cdot \{b \succ a, b \simeq b\}}\,rbrs
$$

Example using Equality Reasoning

$$\dfrac{\dfrac{\dfrac{\begin{array}{c} a \approx b \\ a \approx c \\ \boldsymbol{\neg P(c,b)} \\ \forall x.\ P(c,c) \vee \neg(x \approx c) \end{array}}{P(c,c) \vee \neg(X \approx c)}\,\gamma_{\forall}}{\boldsymbol{P(c,c)} \qquad \neg(X \approx c)}\,\beta_{\vee}}{\dfrac{\langle \{a \approx b, a \approx c\}, b, c\rangle \cdot \emptyset}{\langle \{a \approx b, a \approx c\}, a, c\rangle \cdot \{b \succ a, b \simeq b\}}\,rbrs}$$

Example using Equality Reasoning

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c} a \approx b \\ a \approx c \\ \boldsymbol{\neg P(c,b)} \end{array}}{\forall x.\, P(c,c) \vee \neg(x \approx c)}\,\gamma_\forall}{P(c,c) \vee \neg(X \approx c)}\,\beta_\vee}{\boldsymbol{P(c,c)} \qquad \neg(X \approx c)}
$$

$$
\dfrac{\dfrac{\dfrac{\langle \{a \approx b, a \approx c\}, b, c\rangle \cdot \emptyset}{\langle \{a \approx b, a \approx c\}, a, c\rangle \cdot \{b \succ a, b \simeq b\}}\,rbrs}{\langle \{a \approx b, a \approx c\}, c, c\rangle \cdot \{a \succ c, a \simeq a, b \succ a, b \simeq b\}}\,rbrs
$$

Example using Equality Reasoning

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{a \approx b}
\qquad
\dfrac{}{a \approx c}
}{\neg \boldsymbol{P(c,c)}}
}{\forall x.\, P(c,c) \vee \neg(x \approx c)}\,\gamma_{\forall}
}{P(c,c) \vee \neg(X \approx c)}\,\beta_{\vee}
}{\dfrac{\boldsymbol{P(c,c)}}{\odot}\,\odot \qquad \neg(X \approx c)}
$$

Example using Equality Reasoning

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a \approx b \quad a \approx c \quad \neg P(c,c) \quad \forall x.\, P(c,c) \vee \neg(x \approx c)}{P(c,c) \vee \neg(c \approx c)}\,\gamma_\forall}{P(c,c) \quad \neg(\boldsymbol{X \approx c})}\,\beta_\vee}{\odot}\,\odot}{\odot}$$

$\neg(X \approx c)$

Example using Equality Reasoning

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a \approx b \quad a \approx c \quad \neg P(c,c)}{\forall x.\ P(c,c) \vee \neg(x \approx c)}}{P(c,c) \vee \neg(X \approx c)}\,\gamma_{\forall}}{P(c,c) \quad \neg(\boldsymbol{c} \approx \boldsymbol{c})}\,\beta_{\vee}}{\dfrac{P(c,c)}{\odot}\,\odot \qquad \dfrac{\neg(\boldsymbol{c} \approx \boldsymbol{c})}{\odot}\,\odot}}{\dfrac{\neg(X \approx c)}{\{X \mapsto c\}}\,\odot}$$

Reasoning Modulo Theory

Simple Set Theory

- A_1: $\forall a, b.\ a \subseteq b \Leftrightarrow \forall x.\ x \in a \Rightarrow x \in b$
- A_2: $\forall a, b.\ a = b \Leftrightarrow a \subseteq b \wedge b \subseteq a$
- C: $\forall a.\ a \subseteq a$

$$A_1 \wedge A_2 \wedge \neg C$$

$$\begin{aligned}
&(\forall a, b.\ a \subseteq b \Leftrightarrow \forall x.\ x \in a \Rightarrow x \in b) \\
&\wedge (\forall a, b.\ a = b \Leftrightarrow a \subseteq b \wedge b \subseteq a) \\
&\wedge \neg(\forall a.\ a \subseteq a)
\end{aligned}$$

Reasoning Modulo Theory

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\forall a, b.\ a \subseteq b \Leftrightarrow \forall x.\ x \in a \Rightarrow x \in b) \wedge (\forall a, b.\ a = b \Leftrightarrow a \subseteq b \wedge b \subseteq a) \wedge \neg(\forall a.\ a \subseteq a)}{\forall a, b.\ a \subseteq b \Leftrightarrow \forall x.\ x \in a \Rightarrow x \in b,\ \forall a, b.\ a = b \Leftrightarrow a \subseteq b \wedge b \subseteq a,\ \neg(\forall a.\ a \subseteq a)}\,\alpha_\wedge}{\neg(a \subseteq a)}\,\delta_{\neg\forall}}{\forall b.\ A \subseteq b \Leftrightarrow \forall x.\ x \in A \Rightarrow x \in b}\,\gamma_\forall}{A \subseteq B \Leftrightarrow \forall x.\ x \in A \Rightarrow x \in B}\,\gamma_\forall}{\dfrac{A \subseteq B, x \in A \Rightarrow x \in B}{\sigma = \{A \mapsto a, B \mapsto a\}}\,\odot_\sigma \qquad \dfrac{\dfrac{\dfrac{\dfrac{\neg(A \subseteq B), \neg(\forall x.\ x \in A \Rightarrow x \in B)}{\neg(a \subseteq a), \neg(\forall x.\ x \in a \Rightarrow x \in a)}\,\sigma}{\neg(s \in a \Rightarrow s \in a)}\,\delta_{\neg\forall}}{\neg(s \in a), (s \in a)}\,\alpha_{\neg\Rightarrow}}{\odot}\,\odot}\,\beta_\Leftrightarrow$$

Reasoning Modulo Theory

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\forall a, b.\ a \subseteq b \Leftrightarrow \forall x.\ x \in a \Rightarrow x \in b) \wedge (\forall a, b.\ a = b \Leftrightarrow a \subseteq b \wedge b \subseteq a) \wedge \neg(\forall a.\ a \subseteq a)}{\forall a, b.\ a \subseteq b \Leftrightarrow \forall x.\ x \in a \Rightarrow x \in b,\ \forall a, b.\ a = b \Leftrightarrow a \subseteq b \wedge b \subseteq a,\ \neg(\forall a.\ a \subseteq a)}\ \alpha_{\wedge}}{\neg(a \subseteq a)}\ \delta_{\neg\forall}}{\forall b.\ A \subseteq b \Leftrightarrow \forall x.\ x \in A \Rightarrow x \in b}\ \gamma_{\forall}}{A \subseteq B \Leftrightarrow \forall x.\ x \in A \Rightarrow x \in B}\ \gamma_{\forall}}{\dfrac{A \subseteq B, x \in A \Rightarrow x \in B}{\sigma = \{A \mapsto a, B \mapsto a\}}\ \odot_{\sigma} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\neg(A \subseteq B), \neg(\forall x.\ x \in A \Rightarrow x \in B)}{\neg(a \subseteq a), \neg(\forall x.\ x \in a \Rightarrow x \in a)}\ \sigma}{\neg(s \in a \Rightarrow s \in a)}\ \delta_{\neg\forall}}{\neg(s \in a), (s \in a)}\ \alpha_{\neg\Rightarrow}}{\odot}\ \odot}\ \beta_{\Leftrightarrow}$$

Deduction Modulo Theory (DMT)

Main Heuristic

$(\forall \vec{x}.)\ A \Leftrightarrow F$ where:

- A is an atomic formula
- F is a non-atomic formula

Axiom: $\forall a, b.\ a \subseteq b \Leftrightarrow \forall x.\ x \in a \Rightarrow x \in b$

Rule: $A \subseteq B \rightarrow \forall x.\ x \in A \Rightarrow x \in B$

Axiom: $\forall a, b.\ a = b \Leftrightarrow a \subseteq b \wedge b \subseteq a$

Rule: $A = B \rightarrow A \subseteq B \wedge B \subseteq A$

Deduction Modulo Theory (DMT)

Rewrite Rules

$$A \subseteq B \to \forall x.\, x \in A \Rightarrow x \in B$$

$$A = B \to A \subseteq B \wedge B \subseteq A$$

$$\neg(\forall a.\, a \subseteq a)$$

Deduction Modulo Theory (DMT)

Rewrite Rules

$$A \subseteq B \to \forall x.\, x \in A \Rightarrow x \in B$$

$$A = B \to A \subseteq B \wedge B \subseteq A$$

$$\frac{\neg(\forall a.\, a \subseteq a)}{\neg(a \subseteq a)}\, \delta_{\neg\forall}$$

Deduction Modulo Theory (DMT)

Rewrite Rules

$$A \subseteq B \to \forall x.\, x \in A \Rightarrow x \in B$$

$$A = B \to A \subseteq B \wedge B \subseteq A$$

$$\frac{\neg(\forall a.\, a \subseteq a)}{\neg(a \subseteq a)}\, \delta_{\neg\forall}$$

Deduction Modulo Theory (DMT)

Rewrite Rules

$$A \subseteq B \to \forall x.\, x \in A \Rightarrow x \in B$$

$$A = B \to A \subseteq B \wedge B \subseteq A$$

$$\frac{\dfrac{\neg(\forall a.\, a \subseteq a)}{\neg(a \subseteq a)}\,\delta_{\neg\forall}}{\neg(\forall x.\, x \in a \Rightarrow x \in a)} \to (A \mapsto a, B \mapsto a)$$

Deduction Modulo Theory (DMT)

Rewrite Rules

$$A \subseteq B \to \forall x.\ x \in A \Rightarrow x \in B$$

$$A = B \to A \subseteq B \wedge B \subseteq A$$

$$\frac{\dfrac{\neg(\forall a.\ a \subseteq a)}{\neg(a \subseteq a)}\ \delta_{\neg\forall}}{\neg(\forall x.\ x \in a \Rightarrow x \in a)} \to (A \mapsto a, B \mapsto a)$$

Deduction Modulo Theory (DMT)

Rewrite Rules

$$A \subseteq B \to \forall x.\, x \in A \Rightarrow x \in B$$

$$A = B \to A \subseteq B \wedge B \subseteq A$$

$$\dfrac{\dfrac{\dfrac{\neg(\forall a.\, a \subseteq a)}{\neg(a \subseteq a)}\,\delta_{\neg\forall}}{\neg(\forall x.\, x \in a \Rightarrow x \in a)} \to (A \mapsto a, B \mapsto a)}{\neg(s \in a \Rightarrow s \in a)}\,\delta_{\neg\forall}$$

Deduction Modulo Theory (DMT)

Rewrite Rules

$$A \subseteq B \to \forall x.\, x \in A \Rightarrow x \in B$$

$$A = B \to A \subseteq B \wedge B \subseteq A$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\forall a.\, a \subseteq a)}{\neg(a \subseteq a)}\,\delta_{\neg\forall}}{\neg(\forall x.\, x \in a \Rightarrow x \in a)} \to (A \mapsto a, B \mapsto a)}{\neg(s \in a \Rightarrow s \in a)}\,\delta_{\neg\forall}}{\neg(s \in a), (s \in a)}\,\alpha_{\neg\Rightarrow}$$

Deduction Modulo Theory (DMT)

Rewrite Rules

$$A \subseteq B \to \forall x.\, x \in A \Rightarrow x \in B$$

$$A = B \to A \subseteq B \wedge B \subseteq A$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(\forall a.\, a \subseteq a)}{\neg(a \subseteq a)}\,\delta_{\neg\forall}}{\neg(\forall x.\, x \in a \Rightarrow x \in a)}\,\to (A \mapsto a, B \mapsto a)}{\neg(s \in a \Rightarrow s \in a)}\,\delta_{\neg\forall}}{\neg(s \in a), (s \in a)}\,\alpha_{\neg\Rightarrow}}{\odot}\,\odot$$

Deduction Modulo Theory (DMT)

Benefits

- Avoid combinatorial explosion
- “Useless” axioms aren’t triggered
- Shorter proof
- Not limited to one theory

Integration

- Triggered when a predicate is generated
- Backtrack if multiples rules are available

Scale-Up Experimental Results

	SYN (207 problems)	SET (113 problems)
2	**1.5 s**	**20 s** (+4)
4	**0.6 s**	**15 s** (+5)
8	**0.4 s**	**12 s** (+8)
16	**0.8 s**	**8.7 s** (+10)
28	**0.3 s** (+ 2)	**8.7 s** (+11)

Table 1: Scale-up experimental results of Goéland.

	SYN (207 problems)	SET (208 problems)
2	**1.4 s** (+ 1)	**6.1 s** (+ 5)
4	**1.3 s**	**5.3 s** (+ 8)
8	**1.1 s**	**4.7 s** (+ 7)
16	**0.6 s** (+ 1)	**4.2 s** (+ 9)
28	**0.4 s** (+ 2)	**3.1 s** (+ 9)

Table 2: Scale-up experimental results of Goéland+DMT.

Proof Tree and Segments

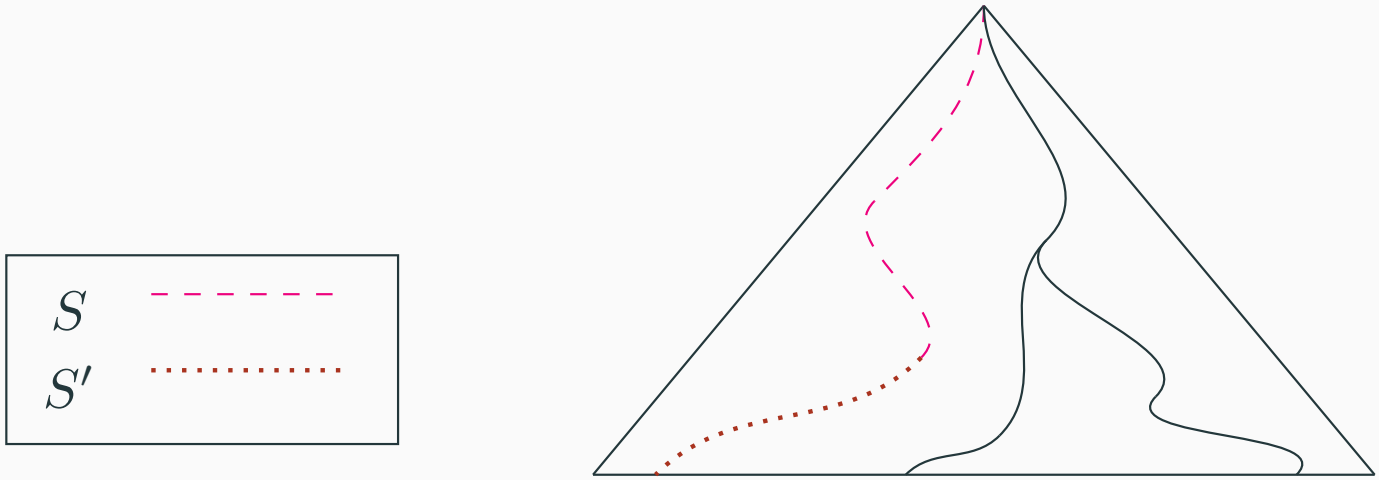


Figure 8: S is an initial segment, S' is a branch, and $S \sqsubseteq S'$.

Mapping

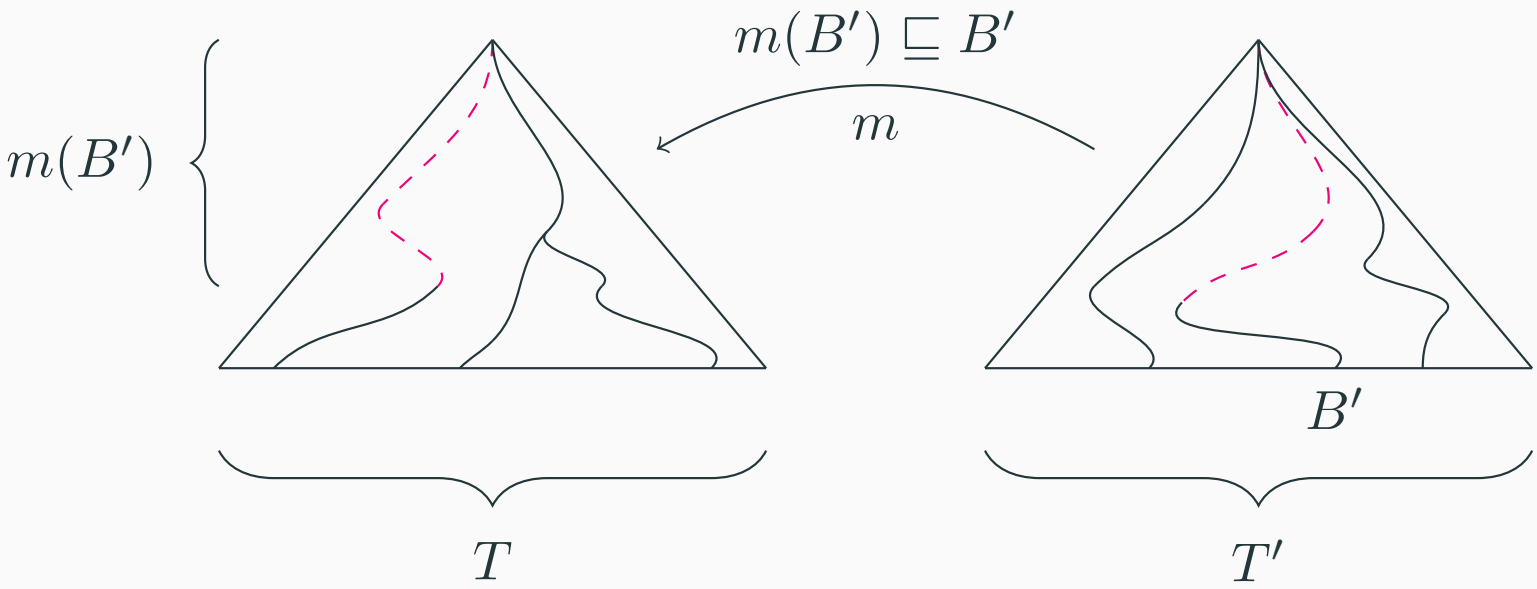


Figure 9: The branch B' is mapped to the initial segment $m(B')$, which means B' contains at least all the formulas of $m(B')$.

Proof

Key Ideas of the Proof

- We consider a proof (T, σ) for a formula F with a reintroduction limit l
- We consider the proof (T', σ') generated by Goéland with the same limit
- We build a mapping between T and T' and show that every branch is T is going to have at least all the formulas than the equivalent one in T

Critical Points

- The agreement mechanism terminates
- A "good" substitution cannot be forbidden