Reasoning methods in Automated Theorem Proving

Presentation at BOREAL team's seminar

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Formal methods and proofs

Formal methods

« Formal method are mathematically rigorous techniques for the specification, development, and verification of software and hardware systems. »

More precisely

- Critical system (life, money)
- Safety by proving ($\neq$ tests)
- But expensive and hard to understand

How to make a proof?

Data

- A language
- Some hypotheses (or not)
- A goal

Inference rules

$$\frac{H1 \qquad H2}{C}$$

How to make a proof?

Data

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Inference rules

$$\frac{A \qquad A \Rightarrow B}{B}$$

Valid and satisifable

Validity

- Always true
- Proof by refutation ($\neg F$ is unsatisfiable)
- Theorem proving

Satisfiability

- True in at least one interpretation
- Building an interpretation
- Constraints solving, find bugs or counter-example

Many possibilites!

Different ways to make a proof

- Hand
- Proof assistant
- Automated theorem prover

Depending of the context

- Valid or satisfiable
- Logic (classical, modal, ...)
- Reasoning inside theories

Logic, expressivity and automation

Decidable Semi-decidable Undecidable

Propositional logic	Fragments of Theories	First-order logic	Higher-order logic	Intuitionist type theory

SAT provers	Decision procedure	SMT provers	First-order provers	Higher-order provers	Interactive proof assistants

Automation Expressiveness

Logic, expressivity and automation

Decidable	Semi-decidable	Undecidable

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Automation	Expressiveness

Many ways to prove, depending of what you get and what you want!

Automated reasoning

Automated reasoning

Automated theorem proving

Given a set of hypotheses and a goal, automatically find a proof!

Reasoning methods in first-order logic

- Saturation based methods
- Tableaux based methods
- Inverse method

Resolution

Context and use

- 1960 by Davis and Putnam
- Saturation based
- Split the original formula into clauses
- Resolve clauses and try to find the empty one

Pros

- Gives the best practical results
- Easy to implement

Cons

- Breaks the initial formula into clauses
- No proof

Example

Formula to prove

$$\neg(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)$$

CNF :

$$\{\neg A\}, \{A \vee A\}, \{\neg B \vee A\}$$

$$\{\neg A\}$$

$$\{A \vee A\}$$

$$\{\neg B \vee A\}$$

Example

Formula to prove

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CNF : $$\{\neg A\}, \{A \vee A\}, \{\neg B \vee A\}$$

$$\begin{tikzcd}
\{\neg A\} \arrow[dr, dash] & \\
\{A \vee A\} \arrow[r, dash] & \{A\} \\
\{\neg B \vee A\} &
\end{tikzcd}$$

Example

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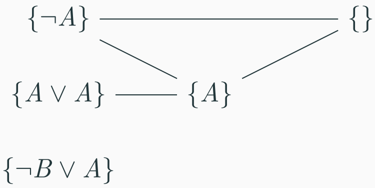

Method of analytics tableaux

Context and use

- 1955 by Beth and Hintikka
- Sequent based
- Tree structure
- Reduce goal to subgoals and try to solve them

Pros

- Gives a proof of the initial formula
- Useful in non-classical logic
- Good match with interactive theorem provers

Cons

- Slower than resolution
- Harder to implement

Example

Formula to prove

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Example

Formula to prove

$$\neg(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)$$

$$\dfrac{\neg(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)}{(A \Rightarrow B) \Rightarrow A, \neg A}\ \alpha_{\neg\Rightarrow}$$

Example

Formula to prove

$$\neg(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)$$

$$\cfrac{\neg(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)}{\cfrac{(A \Rightarrow B) \Rightarrow A, \neg A}{\neg(A \Rightarrow B) \qquad A}\ \beta_{\Rightarrow}}\ \alpha_{\neg\Rightarrow}$$

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Formula to prove

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Inverse method

Context and use

- 1964 by S.Ju. Maslov
- Construct goals from previously proved subgoals
- Use a saturation algorithm
- Forward-chaining proof-search
- Subformula property

Pros

- Gives a proof of the initial formula
- Isomorphic to skolem chase

Cons

- Few implementations
- Slower than resolution and tableaux
- Harder to implement

Example

Formula to prove

$$\neg(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)$$

Rules needed : $\dfrac{\neg P \qquad Q}{P \Rightarrow Q} \Rightarrow$ $\qquad \dfrac{P, \neg Q}{\neg(P \Rightarrow Q)} \neg\Rightarrow$

Available axioms : $\dfrac{}{\Gamma, A, \neg A}\, ax$ $\qquad \dfrac{}{\Gamma, B, \neg B}\, ax$

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To go further

Reasoning with theory

Core provers

- SAT solver
- First-order theorem prover
- ...

Specific provers

- Decision procedure
- Background reasoner
- Equality reasoning

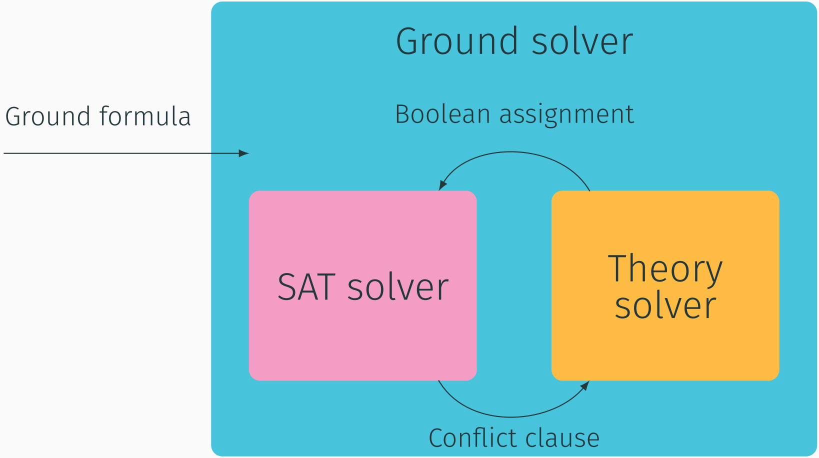
Nesting dolls principle

Expressive logic,
semi-automated tools
(ex : Proof assistants)

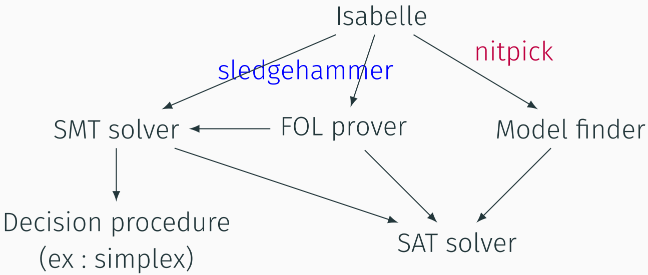
Limited logic,
efficient provers
(ex : SAT)

Intermediate tools
(ex : first-order prover,
background reasoners)

SMT = SAT + Theory solver



Isabelle


Isabelle
sledgehammer
nitpick
SMT solver
FOL prover
Model finder
Decision procedure
(ex : simplex)
SAT solver

Conclusion

Combinations of techniques

- Automated reasoning makes formal method more accessible
- Various methods for various situations
- Cooperation is the key

To go further

- Portfolio approach
- Unification
- Graph algorithms

Thank you for you attention!